A Course Material on

MA – 6351 TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

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MA8353 TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS L T P C 3 1 0 4

OBJECTIVES:

To introduce Fourier series analysis which is central to many applications in engineering apart from its use in solving boundary value problems?

To acquaint the student with Fourier transform techniques used in wide variety of situations.

To introduce the effective mathematical tools for the solutions of partial differential equations that model several physical processes and to develop Z transform techniques for discrete time Systems.

UNIT I PARTIAL DIFFERENTIAL EQUATIONS 9+3

Formation of partial differential equations – Singular integrals -- Solutions of standard types of first order partial differential equations - Lagrange's linear equation -- Linear partial differential equations of second and higher order with constant coefficients of both homogeneous and non-homogeneous types.

UNIT II FOURIER SERIES 9+3

Dirichlet's conditions – General Fourier series – Odd and even functions – Half range sine series –Half range cosine series – Complex form of Fourier series – Parseval's identity – Harmonic analysis.

UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL 9+3

Classification of PDE – Method of separation of variables - Solutions of one dimensional wave equation – One dimensional equation of heat conduction – Steady state solution of two dimensional equation of heat conduction (excluding insulated edges).

UNIT IV FOURIER TRANSFORMS 9+3

Statement of Fourier integral theorem – Fourier transform pair – Fourier sine and cosine transforms – Properties – Transforms of simple functions – Convolution theorem – Parseval's identity.

UNIT V Z - TRANSFORMS AND DIFFERENCE EQUATIONS 9+3

Z- transforms - Elementary properties – Inverse Z - transform (using partial fraction and residues) – Convolution theorem - Formation of difference equations – Solution of difference equations using Z - transform.

TOTAL (L:45+T:15): 60 PERIODS.

TEXT BOOKS:

1. Veerarajan. T., "Transforms and Partial Differential Equations", Tata McGraw Hill Education Pvt. Ltd., New Delhi, Second reprint, 2012.
2. Grewal. B.S., "Higher Engineering Mathematics", 42nd Edition, Khanna Publishers, Delhi, 2012.
3. Narayanan.S., Manicavachagom Pillay.T.K and Ramanaiah.G "Advanced Mathematics for Engineering Students" Vol. II & III, S.Viswanathan Publishers Pvt. Ltd.1998.

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1. Bali.N.P and Manish Goyal, "A Textbook of Engineering Mathematics", 7th Edition, Laxmi Publications Pvt Ltd, 2007.
2. Ramana.B.V., "Higher Engineering Mathematics", Tata Mc Graw Hill Publishing Company Limited, NewDelhi, 2008.
3. Glyn James, "Advanced Modern Engineering Mathematics", 3rd Edition, Pearson Education, 2007.
4. Erwin Kreyszig, "Advanced Engineering Mathematics", 8th Edition, Wiley India, 2007.
5. Ray Wylie. C and Barrett.L.C, "Advanced Engineering Mathematics" Tata Mc Graw Hill Education Pvt Ltd, Sixth Edition, New Delhi, 2012.
6. Datta.K.B., "Mathematical Methods of Science and Engineering", Cengage Learning India Pvt Ltd, Delhi, 2013.

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UNIT– I

PARTIAL DIFFERENTIAL EQUATIONS

This unit covers topics that explain the formation of partial differential equations and the solutions of special types of partial differential equations.

INTRODUCTION

A partial differential equation is one which involves one or more partial derivatives. The order of the highest derivative is called the order of the equation. A partial differential equation contains more than one independent variable. But, here we shall consider partial differential equations involving one dependent variable „z" and only two independent variables x and y so that z = f(x,y). We shall denote

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

A partial differential equation is linear if it is of the first degree in the dependent variable and its partial derivatives. If each term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be homogeneous, otherwise it is non homogeneous.

Formation of Partial Differential Equations

Partial differential equations can be obtained by the elimination of arbitrary constants or by the elimination of arbitrary functions.

By the elimination of arbitrary constants

Let us consider the function

$$\phi (x, y, z, a, b) = 0 \text{-------------- (1)}$$

where a & b are arbitrary constants

Differentiating equation (1) partially w.r.t x & y, we get

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

$$\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Eliminating a and b from equations (1), (2) and (3), we get a partial differential equation of the first order of the form f (x,y,z, p, q) = 0

Example 1

Eliminate the arbitrary constants a & b from $z = ax + by + ab$

Consider $z = ax + by + ab$ ______________(1)

Differentiating (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = a \qquad \text{i.e,} \quad p = a \qquad \_\_\_\_\_\_\_\_\_\_(2)$$

$$\frac{\partial z}{\partial y} = b \qquad \text{i.e,} \quad q = b \qquad \_\_\_\_\_\_\_\_\_\_(3)$$

Using (2) & (3) in (1), we get

$$z = px + qy + pq$$

which is the required partial differential equation.

Example 2

Form the partial differential equation by eliminating the arbitrary constants a and b from
$z = (x^2 + a^2)(y^2 + b^2)$

Given $z = (x^2 + a^2)(y^2 + b^2)$ ____________(1)

Differentiating (1) partially w.r.t x & y , we get

$$p = 2x (y^2 + b^2)$$

$$q = 2y (x + a)$$

Substituting the values of p and q in (1), we get

$$4xyz = pq$$

which is the required partial differential equation.

Example 3

Find the partial differential equation of the family of spheres of radius one whose centre lie in the xy - plane.

The equation of the sphere is given by

$$(x - a)^2 + (y - b)^2 + z^2 = 1 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(1)$$

Differentiating (1) partially w.r.t x & y , we get

$$2(x-a) + 2zp = 0$$
$$2(y-b) + 2zq = 0$$

From these equations we obtain

$$x-a = -zp \text{\_\_\_\_\_\_\_\_\_\_\_}(2)$$
$$y-b = -zq \text{\_\_\_\_\_\_\_\_\_\_\_}(3)$$

Using (2) and (3) in (1), we get

$$z^2p^2 + z^2q^2 + z^2 = 1$$
$$\text{or} \quad z^2(p^2 + q^2 + 1) = 1$$

Example 4

Eliminate the arbitrary constants a, b & c from

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and form the partial differential equation.

The given equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(1)$$

Differentiating (1) partially w.r.t x & y, we get

$$\frac{2x}{a^2} + \frac{2zp}{c^2} = 0$$

$$\frac{2y}{b^2} + \frac{2zq}{c^2} = 0$$

Therefore we get

$$\frac{x}{a^2} + \frac{zp}{c^2} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(2)$$

$$\frac{y}{b^2} + \frac{zq}{c^2} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(3)$$

Again differentiating (2) partially w.r.t „x", we set

$(1/a^2) + (1/c^2)(zr + p^2) = 0$ ________________(4)

Multiplying (4) by x, we get

$$\frac{x}{a^2} + \frac{xz\,r}{c^2} + \frac{p^2x}{c^2} = 0$$

From (2) , we have

$$\frac{-zp}{c^2} + \frac{xzr}{c^2} + \frac{p^2x}{c^2} = 0$$

or $-zp + xzr + p^2x = 0$

By the elimination of arbitrary functions

Let u and v be any two functions of x, y, z and $\Phi(u, v) = 0$, where Φ is an arbitrary function. This relation can be expressed as

$u = f(v)$________________(1)

Differentiating (1) partially w.r.t x & y and eliminating the arbitrary functions from these relations, we get a partial differential equation of the first order of the form
$f(x, y, z, p, q) = 0$.

Example 5

Obtain the partial differential equation by eliminating „f „ from $z = (x+y) f(x^2 - y^2)$

Let us now consider the equation

$$z = (x+y) f(x^2 - y^2) \quad \text{------------} (1)$$

Differentiating (1) partially w.r.t x & y , we get

$$p = (x + y) f'(x^2 - y^2) . 2x + f(x^2 - y^2)$$
$$q = (x + y) f'(x^2 - y^2) . (-2y) + f(x^2 - y^2)$$

These equations can be written as

$$p - f(x^2 - y^2) = (x + y) f'(x^2 - y^2) . 2x \quad \text{------------} (2)$$
$$q - f(x^2 - y^2) = (x + y) f'(x^2 - y^2) .(-2y) \quad \text{------------} (3)$$

Hence, we get

$$\frac{p - f(x^2 - y^2)}{q - f(x^2 - y^2)} = -\frac{x}{y}$$

i.e, $py - yf(x^2 - y^2) = -qx + xf(x^2 - y^2)$

i.e, $py + qx = (x+y) f(x^2 - y^2)$

Therefore, we have by(1), $py + qx = z$

Example 6

Form the partial differential equation by eliminating the arbitrary function f from
$z = e^y f(x + y)$

Consider $z = e^y f(x + y)$ ------------(1)

Differentiating (1) partially w .r. t x & y, we get

$$p = e^y f'(x + y)$$
$$q = e^y f'(x + y) + f(x + y). e^y$$

Hence, we have

$$q = p + z$$

Example 7

Form the PDE by eliminating f & Φ from $z = f(x + ay) + \Phi(x - ay)$

Consider $z = f(x + ay) + \Phi(x - ay)$ ________________(1)

Differentiating (1) partially w.r.t x &y , we get

$p = f'(x + ay) + \Phi'(x - ay)$ ________________(2)

$q = f'(x + ay).a + \Phi'(x - ay)(-a)$________________(3)

Differentiating (2) & (3) again partially w.r.t x & y, we get

$r = f''(x+ay) + \Phi''(x - ay)$
$t = f''(x+ay).a^2 + \Phi''(x - ay)(-a)^2$

i.e, $t = a^2 \{ f''(x + ay) + \Phi''(x - ay)\}$

or $t = a^2 r$

Exercises:

1. Form the partial differential equation by eliminating the arbitrary constants „a" & „b" from the following equations.
   (i) $z = ax + by$
   (ii) $\dfrac{x^2 + y^2}{a^2} + \dfrac{z^2}{b^2} = 1$
   (iii) $z = ax + by + \sqrt{a^2 + b^2}$
   (iv) $ax^2 + by^2 + cz^2 = 1$
   (v) $z = a^2x + b^2y + ab$

2. Find the PDE of the family of spheres of radius 1 having their centres lie on the xy plane{Hint: $(x - a)^2 + (y - b)^2 + z^2 = 1$}

3. Find the PDE of all spheres whose centre lie on the (i) z axis (ii) x-axis

4. Form the partial differential equations by eliminating the arbitrary functions in the following cases.
   (i) $z = f(x + y)$
   (ii) $z = f(x^2 - y^2)$
   (iii) $z = f(x^2 + y^2 + z^2)$
   (iv) $\phi(xyz, x + y + z) = 0$

(v) $z = x + y + f(xy)$

(vi) $z = xy + f(x^2 + y^2)$

(vii) $z = f\left(\frac{xy}{z}\right)$

(viii) $F(xy + z^2, x + y + z) = 0$

(ix) $z = f(x + iy) + f(x - iy)$

(x) $z = f(x^3 + 2y) + g(x^3 - 2y)$

SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

A solution or integral of a partial differential equation is a relation connecting the dependent and the independent variables which satisfies the given differential equation. A partial differential equation can result both from elimination of arbitrary constants and from elimination of arbitrary functions as explained in section 1.2. But, there is a basic difference in the two forms of solutions. A solution containing as many arbitrary constants as there are independent variables is called a complete integral. Here, the partial differential equations contain only two independent variables so that the complete integral will include two constants.A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

Singular Integral

Let $f(x,y,z,p,q) = 0$ -----------(1)

be the partial differential equation whose complete integral is

$\phi(x,y,z,a,b) = 0$ --------------- (2)

where „a" and „b" are arbitrary constants.

Differentiating (2) partially w.r.t. a and b, we obtain

$$\frac{\partial \phi}{\partial a} = 0 \quad \text{----------- (3)}$$

and

$$\frac{\partial \phi}{\partial b} = 0 \quad \text{----------- (4)}$$

The eliminant of „a" and „b" from the equations (2), (3) and (4), when it exists, is called the singular integral of (1).

General Integral

In the complete integral (2), put b = F(a), we get

$$\phi(x,y,z,a,F(a)) = 0 \quad \text{---}(5)$$

Differentiating (2), partially w.r.t.a, we get

$$\frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial b} F'(a) = 0 \quad \text{---------------------------}(6)$$

The eliminant of „a‟ between (5) and (6), if it exists, is called the general integral of (1).

SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS.

The first order partial differential equation can be written as

$$f(x,y,z,p,q) = 0,$$

where $p = \partial z/\partial x$ and $q = \partial z / \partial y$. In this section, we shall solve some standard forms of equations by special methods.

Standard I : f (p,q) = 0. i.e, equations containing p and q only.

Suppose that z = ax + by +c is a solution of the equation f(p,q) = 0, where f (a,b) = 0.

Solving this for b, we get b = F (a).

Hence the complete integral is $z = ax + F(a)\,y + c$ -------------------------(1)

Now, the singular integral is obtained by eliminating a & c between

$$z = ax + y\,F(a) + c$$
$$0 = x + y\,F'(a)$$
$$0 = 1.$$

The last equation being absurd, the singular integral does not exist in this case.

To obtain the general integral, let us take $c = \Phi(a)$.

Then, $z = ax + F(a)\, y + \Phi(a)$ ---------------------- (2)

Differentiating (2) partially w.r.t. a, we get

$$0 = x + F'(a).\, y + \Phi'(a) \quad \text{------------------------(3)}$$

Eliminating „a‟ between (2) and (3), we get the general integral

Example 8

Solve pq = 2

The given equation is of the form f (p,q) = 0

The solution is z = ax + by +c, where ab = 2.

Solving, $b = \dfrac{2}{a}$.

The complete integral is

$$Z = ax + \frac{2}{a} y + c \quad \text{-------------------------------(1)}$$

Differentiating (1) partially w.r.t „c‟, we get

$$0 = 1,$$

which is absurd. Hence, there is no singular integral.

To find the general integral, put c = Φ (a) in (1), we get

$$Z = ax + \frac{2}{a} y + \Phi(a)$$

Differentiating partially w.r.t „a‟, we get

$$0 = x - \frac{2}{a^2} y + \Phi'(a)$$

Eliminating „a‟ between these equations gives the general integral.

Example 9

Solve pq + p +q = 0

The given equation is of the form f (p,q) = 0.

The solution is z = ax + by +c, where ab + a + b = 0.

Solving, we get

$$b = -\frac{a}{1+a}$$

Hence the complete Integral is $z = ax - \frac{a}{1+a} y + c$ ------------------------(1)

Differentiating (1) partially w.r.t. „c", we get

$$0 = 1.$$

The above equation being absurd, there is no singular integral for the given partial differential equation.

To find the general integral, put c = Φ (a) in (1), we have

$$z = ax - \frac{a}{1+a} y + \Phi(a) \text{ -------------------------------------- (2)}$$

Differentiating (2) partially w.r.t a, we get

$$0 = x - \frac{1}{(1+a)^2} y + \Phi'(a) \text{ --------------------------------------(3)}$$

Eliminating „a" between (2) and (3) gives the general integral.

Example 10

Solve $p^2 + q^2 = npq$

The solution of this equation is z = ax + by + c, where $a^2 + b^2 = nab$.

Solving, we get

$$b = a \frac{n \pm \sqrt{(n^2 - 4)}}{\quad}$$

2

Hence the complete integral is

$$z = ax + a\left(\frac{n \pm \sqrt{n^2-4}}{2}\right) y + c \qquad \text{------------------------(1)}$$

Differentiating (1) partially w.r.t c, we get 0 = 1, which is absurd. Therefore, there is no singular integral for the given equation.

To find the general Integral, put C = Φ (a), we get

$$z = ax + a\frac{n + \sqrt{n^2-4}}{2} y + \Phi(a)$$

Differentiating partially w.r.t „a", we have

$$0 = x + \frac{n \pm \sqrt{n^2-4}}{2} y + \Phi'(a)$$

The eliminant of „a" between these equations gives the general integral

Standard II : Equations of the form f (x,p,q) = 0, f (y,p,q) = 0 and f (z,p,q) = 0.
i.e, one of the variables x,y,z occurs explicitly.

(i) Let us consider the equation f (x,p,q) = 0.

Since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or $dz = pdx + qdy$

Assume that q = a.

Then the given equation takes the form f (x, p,a) = 0

Solving, we get $p = \Phi(x,a)$.

Therefore, $dz = \Phi(x,a)\, dx + a\, dy$.

Integrating, $z = \int \Phi(x,a)\, dx + ay + b$ which is a complete Integral.

(ii) Let us consider the equation $f(y,p,q) = 0$.

Assume that $p = a$.

Then the equation becomes $f(y,a,q) = 0$

Solving, we get $q = \Phi(y,a)$.

Therefore, $dz = a\,dx + \Phi(y,a)\,dy$.

Integrating, $z = ax + \int \Phi(y,a)\,dy + b$, which is a complete Integral.

(iii) Let us consider the equation $f(z, p, q) = 0$.

Assume that $q = ap$.

Then the equation becomes $f(z, p, ap) = 0$

Solving, we get $p = \Phi(z,a)$. Hence $dz = \Phi(z,a)\,dx + a\,\Phi(z,a)\,dy$.

$$\text{ie, } \frac{dz}{\Phi(z,a)} = dx + a\,dy.$$

Integrating, $$\int \frac{dz}{\Phi(z,a)} = x + ay + b,$$ which is a complete Integral.

Example 11

Solve $q = xp + p^2$

Given $q = xp + p^2$ -------------- (1)

This is of the form $f(x,p,q) = 0$.

Put $q = a$ in (1), we get

$$a = xp + p^2$$

i. e, $p^2 + xp - a = 0$.

Therefore, $$p = \frac{-x + \sqrt{(x^2 + 4a)}}{2}$$

Integrating , $z = \int \frac{-x \pm \sqrt{x^2 + 4a}}{2} dx + ay + b$

Thus, $z = -\frac{x^2}{4} \pm \left\{ \frac{x}{4} \sqrt{(4a + x^2)} + a \sin h^{-1} \frac{x}{2\sqrt{a}} \right\} + ay + b$

Example 12

Solve $q = yp^2$

This is of the form f (y,p,q) = 0

Then, put p = a.

Therfore, the given equation becomes $q = a^2y$.

Since dz = pdx + qdy, we have

$dz = adx + a^2y\, dy$

Integrating, we get $z = ax + \frac{a^2y^2}{2} + b$

Example 13

Solve $9 (p^2z + q^2) = 4$

This is of the form f (z,p,q) = 0

Then, putting q = ap, the given equation becomes

$9 (p^2z + a^2p^2) = 4$

Therefore, $p = \pm \frac{2}{3 (\sqrt{z + a^2})}$

and $q = \pm \frac{2a}{3 (\sqrt{z + a^2})}$

Since dz = pdx + qdy,

$$dz = \pm \frac{2}{3} \frac{1}{\sqrt{z + a^2}} dx \pm \frac{2}{3} a \frac{1}{\sqrt{z + a^2}} dy$$

Multiplying both sides by $\sqrt{z + a^2}$, we get

$$\sqrt{z + a^2}\, dz = \frac{2}{3} dx + \frac{2}{3} a\, dy$$, which on integration gives,

$$\frac{(z+a^2)^{3/2}}{3/2} = \frac{2}{3} x + \frac{2}{3} ay + b.$$

or $$(z + a^2)^{3/2} = x + ay + b.$$

Standard III : $f_1(x,p) = f_2(y,q)$. ie, equations in which 'z' is absent and the variables are separable.

Let us assume as a trivial solution that

$$f(x,p) = g(y,q) = a \text{ (say)}.$$

Solving for p and q, we get $p = F(x,a)$ and $q = G(y,a)$.

But $$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Hence $dz = pdx + qdy = F(x,a)\, dx + G(y,a)\, dy$

Therefore, $z = \int F(x,a)\, dx + \int G(y,a)\, dy + b$, which is the complete integral of the given equation containing two constants a and b. The singular and general integrals are found in the usual way.

Example 14

Solve $pq = xy$

The given equation can be written as

$$\frac{p}{x} = \frac{y}{q} = a \text{ (say)}$$

$$\text{Therefore,} \quad \frac{p}{x} = a \quad \text{implies} \quad p = ax$$

$$\text{and} \quad \frac{y}{q} = a \quad \text{implies} \quad q = \frac{y}{a}$$

Since dz = pdx + qdy, we have

$$dz = ax\,dx + \frac{y}{a}\,dy, \text{ which on integration gives.}$$

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + b$$

Example 15

Solve $p^2 + q^2 = x^2 + y^2$

The given equation can be written as

$$p^2 - x^2 = y^2 - q^2 = a^2 \text{ (say)}$$

$$p^2 - x^2 = a^2 \quad \text{implies } p = \sqrt{(a^2 + x^2)}$$

$$\text{and} \quad y^2 - q^2 = a^2 \quad \text{implies } q = \sqrt{(y^2 - a^2)}$$

But $dz = pdx + qdy$

$$\text{ie, } dz = \sqrt{a^2 + x^2}\,dx + \sqrt{y^2 - a^2}\,dy$$

Integrating, we get

$$z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + b$$

Standard IV (Clairaut's form)

Equation of the type z = px + qy + f (p,q) ------(1) is known as Clairaut"s form.

Differentiating (1) partially w.r.t x and y, we get

$$p = a \quad \text{and} \quad q = b.$$

Therefore, the complete integral is given by

$$z = ax + by + f(a,b).$$

Example 16

Solve $z = px + qy + pq$

The given equation is in Clairaut‟s form.

Putting $p = a$ and $q = b$, we have

$$z = ax + by + ab \text{ --(1)}$$

which is the complete integral.

To find the singular integral, differentiating (1) partially w.r.t a and b, we get

$$0 = x + b$$
$$0 = y + a$$

Therefore we have, $a = -y$ and $b = -x$.

Substituting the values of a & b in (1), we get

$$z = -xy - xy + xy$$

or $\quad z + xy = 0$, which is the singular integral.

To get the general integral, put $b = \Phi(a)$ in (1).

Then $\quad z = ax + \Phi(a)y + a\,\Phi(a)$ --(2)

Differentiating (2) partially w.r.t a, we have

$$0 = x + \Phi'(a)\,y + a\Phi'(a) + \Phi(a) \text{ -----------------------------------(3)}$$

Eliminating „a‟ between (2) and (3), we get the general integral.

Example 17

Find the complete and singular solutions of $z = px + qy + \sqrt{1 + p^2 + q^2}$

The complete integral is given by

$$z = ax + by + \sqrt{1 + a^2 + b^2} \quad \text{--------} (1)$$

To obtain the singular integral, differentiating (1) partially w.r.t a & b. Then,

$$0 = x + \frac{a}{\sqrt{1 + a^2 + b^2}}$$

$$0 = y + \frac{b}{\sqrt{1 + a^2 + b^2}}$$

Therefore,

$$x = \frac{-a}{\sqrt{(1 + a^2 + b^2)}} \quad \text{--------} (2)$$

and

$$y = \frac{-b}{\sqrt{(1 + a^2 + b^2)}} \quad \text{--------} (3)$$

Squaring (2) & (3) and adding, we get

$$x^2 + y^2 = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

Now, $$1 - x^2 - y^2 = \frac{1}{1 + a^2 + b^2}$$

i.e, $$1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2}$$

Therefore,

$$\sqrt{(1 + a^2 + b^2)} = \frac{1}{\sqrt{1 - x^2 - y^2}} \quad \text{--------} (4)$$

Using (4) in (2) & (3), we get

$$x = -a\sqrt{1 - x^2 - y^2}$$

and $$y = -b\sqrt{1 - x^2 - y^2}$$

Hence, $$a = \frac{-x}{\sqrt{1-x^2-y^2}} \quad \text{and} \quad b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Substituting the values of a & b in (1) , we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

which on simplification gives

$$z = \sqrt{1 - x^2 - y^2}$$

or $x^2 + y^2 + z^2 = 1$, which is the singular integral.

Exercises

Solve the following Equations

1. $pq = k$
2. $p + q = pq$
3. $\sqrt{p} + \sqrt{q} = x$
4. $p = y^2q^2$
5. $z = p^2 + q^2$
6. $p + q = x + y$
7. $p^2z^2 + q^2 = 1$
8. $z = px + qy - 2\sqrt{pq}$
9. $\{z - (px + qy)\}^2 = c^2 + p^2 + q^2$
10. $z = px + qy + p^2q^2$

EQUATIONS REDUCIBLE TO THE STANDARD FORMS

Sometimes, it is possible to have non – linear partial differential equations of the first order which do not belong to any of the four standard forms discussed earlier. By changing the variables suitably, we will reduce them into any one of the four standard forms.

Type (i) : Equations of the form $F(x^m p, y^nq) = 0$ (or) $F(z, x^mp, y^nq) = 0$.

Case(i) : If $m \neq 1$ and $n \neq 1$, then put $x^{1-m} = X$ and $y^{1-n} = Y$.

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = \dfrac{\partial z}{\partial X}(1-m)\, x^{-m}$

Therefore, $x^m p = \dfrac{\partial z}{\partial X}(1-m) = (1-m)P$, where $P = \dfrac{\partial z}{\partial X}$

Similarly, $y^n q = (1-n)Q$, where $Q = \dfrac{\partial z}{\partial Y}$

Hence, the given equation takes the form F(P,Q) = 0 (or) F(z,P,Q) = 0.
Case(ii) : If m = 1 and n = 1, then put log x = X and log y = Y.

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = \dfrac{\partial z}{\partial X}\, \dfrac{1}{x}$

Therefore, $xp = \dfrac{\partial z}{\partial X} = P$.

Similarly, yq =Q.

Example 18

Solve $x^4p^2 + y^2zq = 2z^2$

The given equation can be expressed as

$(x^2p)^2 + (y^2q)z = 2z^2$

Here m = 2, n = 2

Put $X = x^{1-m} = x^{-1}$ and $Y = y^{1-n} = y^{-1}$.

We have $x^m p = (1-m)P$ and $y^n q = (1-n)Q$
i.e, $x^2p = -P$ and $y^2q = -Q$.

Hence the given equation becomes

$P^2 - Qz = 2z^2$ --------------------(1)

This equation is of the form f (z,P,Q) = 0.

Let us take Q = aP.

Then equation (1) reduces to

$$P^2 - aPz = 2z^2$$

Hence, $$P = \frac{a \pm \sqrt{(a^2 + 8)}}{2} z$$

and $$Q = a \frac{a \pm \sqrt{(a^2 + 8)}}{2} z$$

Since dz = PdX + QdY, we have

$$dz = \frac{a \pm \sqrt{(a^2 + 8)}}{2} z\,dX + a \frac{a \pm \sqrt{(a^2 + 8)}}{2} z\,dY$$

i.e, $$\frac{dz}{z} = \frac{a \pm \sqrt{(a^2 + 8)}}{2} (dX + a\,dY)$$

Integrating, we get

$$\log z = \left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)(X + aY) + b$$

Therefore, $$\log z = \frac{a \pm \sqrt{(a^2 + 8)}}{2} \quad \frac{1}{x} + \frac{a}{y} + b$$ which is the complete solution.

Example 19

Solve $x^2p^2 + y^2q^2 = z^2$

The given equation can be written as

$$(xp)^2 + (yq)^2 = z^2$$

Here m = 1, n = 1.

Put X = log x and Y = log y.

Then xp = P and yq = Q.

Hence the given equation becomes

$$P^2 + Q^2 = z^2 \quad \text{--------------} (1)$$

This equation is of the form F(z,P,Q) = 0.

Therefore, let us assume that Q = aP.

Now, equation (1) becomes,

$$P^2 + a^2 P^2 = z^2$$

Hence $$P = \frac{z}{\sqrt{(1+a^2)}}$$

and $$Q = \frac{az}{\sqrt{(1+a^2)}}$$

Since dz = PdX + QdY, we have

$$dz = \frac{z}{\sqrt{(1+a^2)}}dX + \frac{az}{\sqrt{(1+a^2)}} dY.$$

i.e, $$\sqrt{(1+a^2)} \frac{dz}{z} = dX + a\, dY.$$

Integrating, we get

$$\sqrt{(1+a^2)} \log z = X + aY + b.$$

Therefore, $\sqrt{(1+a^2)} \log z = \log x + a\log y + b$, which is the complete solution.

Type (ii) : Equations of the form $F(z^k p, z^k q) = 0$ (or) $F(x, z^k p) = G(y,z^k q)$.

Case (i) : If $k \neq -1$, put $Z = z^{k+1}$,

Now $$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \frac{\partial z}{\partial x} = (k+1)z^k . \frac{\partial z}{\partial x} = (k+1) z^k p.$$

Therefore, $$z^k p = \frac{1}{k+1} \frac{\partial Z}{\partial x}$$

Similarly, $z^k q = \dfrac{1}{k+1} \dfrac{\partial Z}{\partial y}$

Case (ii) : If k = -1, put Z = log z.

Now, $\dfrac{\partial Z}{\partial x} = \dfrac{\partial Z}{\partial z} \dfrac{\partial z}{\partial x} = \dfrac{1}{z} p$

Similarly, $\dfrac{\partial Z}{\partial y} = \dfrac{1}{z} q.$

Example 20

Solve $z^4q^2 – z^2p = 1$

The given equation can also be written as

$$(z^2q)^2 – (z^2p) = 1$$

Here k = 2. Putting $Z = z^{k+1} = z^3$, we get

$$Z^k p = \frac{1}{k+1} \frac{\partial Z}{\partial x} \quad \text{and} \quad Z^k q = \frac{1}{k+1} \frac{\partial Z}{\partial y}$$

i.e, $Z^2 p = \dfrac{1}{3} \dfrac{\partial Z}{\partial x}$ and $Z^2 q = \dfrac{1}{3} \dfrac{\partial Z}{\partial y}$

Hence the given equation reduces to

$$\left(\frac{Q}{3}\right)^2 - \left(\frac{P}{3}\right) = 1$$

i.e, $Q^2 – 3P – 9 = 0,$

which is of the form F(P,Q) = 0.

Hence its solution is Z = ax + by + c, where $b^2 – 3a – 9 = 0$.

Solving for b, $b = \pm \sqrt{(3a + 9)}$

Hence the complete solution is

$$Z = ax \pm \sqrt{(3a + 9)} \,.\, y + c$$

or $$z^3 = ax \pm \sqrt{(3a + 9)}\, y + c$$

Exercises

Solve the following equations.

1. $x^2p^2 + y^2p^2 = z^2$
2. $z^2 (p^2+q^2) = x^2 + y^2$
3. $z^2 (p^2x^2 + q^2) = 1$
4. $2x^4p^2 – yzq – 3z^2 = 0$
5. $p^2 + x^2y^2q^2 = x^2 z^2$
6. $x^2p + y^2q = z^2$
7. $x^2/p + y^2/q = z$
8. $z^2 (p^2 – q^2) = 1$
9. $z^2 (p^2/x^2 + q^2/y^2) = 1$
10. $p^2x + q^2y = z$.

Lagrange's Linear Equation

Equations of the form Pp + Qq = R__________(1), where P, Q and R are functions of x, y, z, are known as Lagrange‟s equations and are linear in „p‟ and „q‟.To solve this equation, let us consider the equations u = a and v = b, where a, b are arbitrary constants and u, v are functions of x, y, z.

Since „u ‟ is a constant, we have du = 0 -----------(2).

But „u‟ as a function of x, y, z,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Comparing (2) and (3), we have

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)$$

Similarly,
$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_ (4)$$

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}} = \frac{dz}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}$$

(or)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(5)}$$

Equations (5) represent a pair of simultaneous equations which are of the first order and of first degree.Therefore, the two solutions of (5) are u = a and v = b. Thus, ϕ(u, v) = 0 is the required solution of (1).

Note :

To solve the Lagrange‟s equation,we have to form the subsidiary or auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which can be solved either by the method of grouping or by the method of multipliers.

Example 21

Find the general solution of px + qy = z**.**

Here, the subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking the first two ratios , $\frac{dx}{x} = \frac{dy}{y}$

Integrating, log x = log y + log c_1

or $x = c_1 y$

i.e, $c_1 = x / y$

From the last two ratios, $\frac{dy}{y} = \frac{dz}{z}$

Integrating, $\log y = \log z + \log c_2$

or $y = c_2 z$

i.e, $c_2 = y / z$

Hence the required general solution is

$\Phi(x/y, y/z) = 0$, where Φ is arbitrary

Example 22

Solve $p \tan x + q \tan y = \tan z$

The subsidiary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking the first two ratios , $\frac{dx}{\tan x} = \frac{dy}{\tan y}$

ie, $\cot x \, dx = \cot y \, dy$

Integrating, $\log \sin x = \log \sin y + \log c_1$

ie, $\sin x = c_1 \sin y$

Therefore, $c_1 = \sin x / \sin y$

Similarly, from the last two ratios, we get

$\sin y = c_2 \sin z$

i.e, $c_2 = \sin y / \sin z$

Hence the general solution is

$$\Phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0, \text{ where } \Phi \text{ is arbitrary.}$$

Example 23

Solve (y-z) p + (z-x) q = x-y

Here the subsidiary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

Using multipliers 1,1,1,

$$\text{each ratio} = \frac{dx + dy + dz}{0}$$

Therefore, dx + dy + dz =0.

Integrating, $x + y + z = c_1$ _____________ (1)

Again using multipliers x, y and z,

$$\text{each ratio} = \frac{xdx + ydy + zdz}{0}$$

Therefore, xdx + ydy + zdz = 0.

Integrating, $x^2/2 + y^2/2 + z^2/2 = \text{constant}$

or $x^2 + y^2 + z^2 = c_2$ ____________(2)

Hence from (1) and (2), the general solution is

$$\Phi(x + y + z, x^2 + y^2 + z^2) = 0$$

Example 24

Find the general solution of (mz - ny) p + (nx- lz)q = ly - mx.

Here the subsidiary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using the multipliers x, y and z, we get

$$\text{each fraction} = \frac{xdx + ydy + zdz}{0}$$

` $\therefore$ xdx + ydy + zdz = 0, which on integration gives

$$x^2/2 + y^2/2 + z^2/2 = \text{constant}$$

or $x^2 + y^2 + z^2 = c_1$ ____________(1)

Again using the multipliers l, m and n, we have

$$\text{each fraction} = \frac{ldx + mdy + ndz}{0}$$

` $\therefore$ ldx + mdy + ndz = 0, which on integration gives

$lx + my + nz = c_2$ ____________(2)

Hence, the required general solution is

$$\Phi(x^2 + y^2 + z^2, lx + my + nz) = 0$$

Example 25

Solve $(x^2 - y^2 - z^2)\, p + 2xy\, q = 2xz$.

The subsidiary equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking the last two ratios,

$$\frac{dx}{2xy} = \frac{dz}{2xz}$$

$$2xy \qquad 2xz$$

ie, $$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $\log y = \log z + \log c_1$

or $y = c_1 z$

i.e, $c_1 = y/z$ ____________(1)

Using multipliers x, y and z, we get

$$\text{each fraction} = \frac{xdx + y\,dy + zdz}{x(x^2-y^2-z^2)+2xy^2+2xz^2} = \frac{xdx + y\,dy + zdz}{x(x^2+y^2+z^2)}$$

Comparing with the last ratio, we get

$$\frac{xdx + y\,dy + zdz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz}$$

i.e, $$\frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2} = \frac{dz}{z}$$

Integrating, $\log(x^2+y^2+z^2) = \log z + \log c_2$

or $x^2+y^2+z^2 = c_2 z$

i.e, $$c_2 = \frac{x^2+y^2+z^2}{z}$$ ____________(2)

From (1) and (2), the general solution is $\Phi(c_1, c_2) = 0$.

i.e, $$\Phi\left((y/z), \frac{x^2+y^2+z^2}{z}\right) = 0$$

Exercises

Solve the following equations

1. $px^2 + qy^2 = z^2$
2. $pyz + qzx = xy$
3. $xp – yq = y^2 – x^2$
4. $y^2zp + x^2zq = y^2x$
5. $z (x – y) = px^2 – qy^2$
6. $(a – x) p + (b – y) q = c – z$
7. $(y^2z p) /x + xzq = y^2$
8. $(y^2 + z^2) p – xyq + xz = 0$
9. $x^2p + y^2q = (x + y) z$
10. $p – q = \log (x+y)$
11. $(xz + yz)p + (xz – yz)q = x^2 + y^2$
12. $(y – z)p – (2x + y)q = 2x + z$

PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS.

Homogeneous Linear Equations with constant Coefficients.

A homogeneous linear partial differential equation of the n^{th} order is of the form

$$c_0 \frac{\partial^n z}{\partial x^n} + c_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots\ldots + c_n \frac{\partial^n z}{\partial y^n} = F (x,y) \text{-------------------}(1)$$

where $c_0, c_1,$---------, c_n are constants and F is a function of „x‟ and „y‟. It is homogeneous because all its terms contain derivatives of the same order.

Equation (1) can be expressed as

$$(c_0D^n + c_1D^{n-1} D' + \ldots.. + c_n D'^n) z = F (x,y)$$

or $$f (D,D') z = F (x,y) \text{-------------------}(2),$$

where, $\frac{\partial}{\partial x} \equiv D$ and $\frac{\partial}{\partial y} \equiv D'$.

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely, the complementary function and the particular integral.

The complementary function is the complete solution of $f(D,D')\,z = 0$ ------ (3), which must contain n arbitrary functions as the degree of the polynomial $f(D,D')$. The particular integral is the particular solution of equation (2).

Finding the complementary function

Let us now consider the equation $f(D,D')\,z = F(x,y)$

The auxiliary equation of (3) is obtained by replacing D by m and D' by 1.

i.e, $c_0 m^n + c_1 m^{n-1} + \ldots + c_n = 0$ ------------ (4)

Solving equation (4) for „m", we get „n" roots. Depending upon the nature of the roots, the Complementary function is written as given below:

Roots of the auxiliary equation	Nature of the roots	Complementary function(C.F)
$m_1, m_2, m_3 \ldots\ldots, m_n$	distinct roots	$f_1(y+m_1x)+f_2(y+m_2x) + \ldots\ldots + f_n(y+m_nx)$.
$m_1 = m_2 = m, m_3, m_4, \ldots, m_n$	two equal roots	$f_1(y+m_1x)+xf_2(y+m_1x) + f_3(y+m_3x) + \ldots + f_n(y+m_nx)$.
$m_1 = m_2 = \ldots\ldots = m_n = m$	all equal roots	$f_1(y+mx)+xf_2(y+mx) + x^2f_3(y+mx)+\ldots + \ldots + x^{n-1} f_n(y+mx)$

Finding the particular Integral

Consider the equation $f(D,D')\,z = F(x,y)$.

Now, the P.I is given by $\frac{1}{f(D,D')} F(x,y)$

Case (i) : When $F(x,y) = e^{ax+by}$

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

Replacing D by „a" and D' by „b", we have

$$P.I = \frac{1}{} e^{ax+by}, \quad \text{where } f(a,b) \neq 0.$$

$$\frac{}{f(a,b)}$$

Case (ii) : When F(x,y) = sin(ax + by) (or) cos (ax +by)

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

Replacing $D^2 = -a^2$, $DD'^{\,2} = -ab$ and $D' = -b^2$, we get

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by), \text{ where } f(-a^2, -ab, -b^2) \neq 0.$$

Case (iii) : When $F(x,y) = x^m y^n$,

$$P.I = \frac{1}{f(D,D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

Case (iv) : When F(x,y) is any function of x and y.

$$P.I = \frac{1}{f(D,D')} F(x,y).$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions considering f (D,D') as a function of D alone.

Then operate each partial fraction on F(x,y) in such a way that

$$\frac{1}{D-mD'} F(x,y) = \int F(x,c-mx)\,dx,$$

where c is replaced by y+mx after integration

Example 26

Solve $(D^3 - 3D^2D' + 4D'^3) z = e^{x+2y}$

The auxillary equation is $m = m^3 - 3m^2 + 4 = 0$

The roots are m = -1,2,2

Therefore the C.F is $f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$.

$$\text{P.I.} = \frac{e^{x+2y}}{D^3 - 3D^2D' + 4D'^3} \text{ (Replace D by 1 and D' by 2)}$$

$$= \frac{e^{x+2y}}{1-3(1)(2)+4(2)^3}$$

$$= \frac{e^{x+2y}}{27}$$

Hence, the solution is z = C.F. + P.I

ie, $$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$$

Example 27

Solve $(D^2 - 4DD' + 4D'^2) z = \cos(x-2y)$

The auxiliary equation is $m^2 - 4m + 4 = 0$

Solving, we get m = 2,2.

Therefore the C.F is $f_1(y+2x) + xf_2(y+2x)$.

$$\therefore \text{P.I} = \frac{1}{D^2 - 4DD' + 4D'^2} \cos(x-2y)$$

Replacing D^2 by – 1, DD' by 2 and D'^2 by –4, we have

$$\text{P.I} = \frac{1}{(-1) - 4(2) + 4(-4)} \cos(x-2y)$$

$$= -\frac{\cos(x-2y)}{25}$$

$\therefore$ Solution is $z = f_1(y+2x) + xf_2(y+2x)$ ––––––––––––––– -- .
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Example 28

Solve $(D^2 - 2DD')\,z = x^3y + e^{5x}$

The auxiliary equation is $m^2 - 2m = 0$.

Solving, we get $m = 0,2$.
Hence the C.F is $f_1(y) + f_2(y+2x)$.

$$P.I_1 = \frac{x^3y}{D^2 - 2DD'}$$

$$= \frac{1}{D^2\left(1 - \frac{2D'}{D}\right)}(x^3y)$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1}(x^3y)$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \ldots\ldots\ldots\ldots\right)(x^3y)$$

$$= \frac{1}{D^2}(x^3y) + \frac{2}{D}D'(x^3y) + \frac{4}{D^2}D'^2(x^3y) + \ldots\ldots$$

$$= \frac{1}{D^2}(x^3y) + \frac{2}{D}(x^3) + \frac{4}{D^2}(0) + \ldots\ldots$$

$$P.I_1 = \frac{1}{D^2}(x^3y) + \frac{2}{D^3}(x^3)$$

$$P.I_1 = \frac{x^5y}{20} + \frac{x^6}{60}$$

$$P.I_2 = \frac{e^{5x}}{D^2 - 2DD'} \text{ (Replace D by 5 and } D' \text{ by 0)}$$

$$= \frac{e^{5x}}{25}$$

$$\therefore \text{Solution is } Z = f_1(y) + f_2(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60} + \frac{e^{5x}}{25}$$

Example 29

Solve $(D^2 + DD' - 6 D'^2)\, z = y \cos x$.

The auxiliary equation is $m^2 + m - 6 = 0$.
Therefore, $m = -3, 2$.

Hence the C.F is $f_1(y-3x) + f_2(y + 2x)$.

$$P.I = \frac{y \cos x}{D^2 + DD' - 6D'^2}$$

$$= \frac{y \cos x}{(D + 3D')(D - 2D')}$$

$$= \frac{1}{(D+3D')} \frac{1}{(D - 2D')} y \cos x$$

$$= \frac{1}{(D+3D')} \int (c - 2x) \cos x \, dx, \text{ where } y = c - 2x$$

$$= \frac{1}{(D+3D')} \int (c - 2x)\, d(\sin x)$$

$$= \frac{1}{(D+3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)]$$

$$= \frac{1}{(D+3D')} [y \sin x - 2 \cos x)]$$

$$= \int [(c + 3x) \sin x - 2 \cos x]\, dx, \text{ where } y = c + 3x$$

$$= \int (c + 3x)\, d(-\cos x) - 2\int \cos x \; dx$$

$$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2\sin x$$

$$= -y\cos x + \sin x$$

Hence the complete solution is

$$z = f_1(y - 3x) + f_2(y + 2x) - y\cos x + \sin x$$

Example 30

Solve $r - 4s + 4t = e^{2x+y}$

Given equation is $$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

i.e, $$(D^2 - 4DD' + 4D'^2)\, z = e^{2x+y}$$

The auxiliary equation is $m^2 - 4m + 4 = 0$.

Therefore, $m = 2,2$

Hence the C.F is $f_1(y + 2x) + x f_2(y + 2x)$.

$$\text{P.I.} = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2}$$

Since $D^2 - 4DD' + 4D'^2 = 0$ for $D = 2$ and $D' = 1$, we have to apply the general rule.

$$\therefore \text{P.I.} = \frac{e^{2x+y}}{(D - 2D')(D - 2D')}$$

$$= \frac{1}{(D - 2D')}\,\frac{1}{(D - 2D')}\, e^{2x+y}$$

$$= \frac{1}{(D - 2D')} \int e^{2x+c-2x}\, dx, \text{ where } y = c - 2x.$$

$$= \frac{1}{(D-2D')} \int e_c \, dx$$

$$= \frac{1}{(D-2D')} e^c .x$$

$$= \frac{1}{D-2D'} xe^{y+2x}$$

$= \int xe^{c-2x+2x} \, dx$, where y = c – 2x.

$= \int xe^c \, dx$

$= e^c . x^2/2$

$$= \frac{x^2e^{y+2x}}{2}$$

Hence the complete solution is

$$z = f_1(y+2x) + f_2(y+2x) + \frac{1}{2} x^2e^{2x+y}$$

Non – Homogeneous Linear Equations

Let us consider the partial differential equation

$$f(D,D')\, z = F(x,y) \text{ ---------------- } (1)$$

If $f(D,D')$ is not homogeneous, then (1) is a non–homogeneous linear partial differential equation. Here also, the complete solution = C.F + P.I.

The methods for finding the Particular Integrals are the same as those for homogeneous linear equations.

But for finding the C.F, we have to factorize $f(D,D')$ into factors of the form $D - mD' - c$.

Consider now the equation

$$(D - mD' - c)\, z = 0 \text{ -----------}(2).$$

This equation can be expressed as

$$p - mq = cz \text{ -------- } (3),$$

which is in Lagrangian form.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz} \text{ -----------}(4)$$

The solutions of (4) are $y + mx = a$ and $z = be^{cx}$.

Taking $b = f(a)$, we get $z = e^{cx} f(y+mx)$ as the solution of (2).

Note:

1. If $(D-m_1D' - C_1)(D - m_2D'-C_2) \ldots\ldots (D - m_nD'-C_n) z = 0$ is the partial differential equation, then its complete solution is

   $z = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y+m_2x) + \ldots\ldots\ldots + e^{c_nx} f_n(y+m_nx)$

2. In the case of repeated factors, the equation $(D-mD' - C)^n z = 0$ has a complete

   solution $z = e^{cx} f_1(y + mx) + x\, e^{cx} f_2(y+mx) + \ldots\ldots\ldots + x^{n-1} e^{cx} f_n(y+mx)$.

Example 31

Solve $(D-D'-1)(D-D' - 2)z = e^{2x - y}$

Here $m_1 = 1$, $m_2 = 1$, $c_1 = 1$, $c_2 = 2$.

Therefore, the C.F is $e^x f_1(y+x) + e^{2x} f_2(y+x)$.

$$\text{P.I.} = \frac{e^{2x-y}}{(D - D' - 1)(D-D' - 2)} \quad \text{Put } D = 2,\ D' = -1.$$

$$= \frac{e^{2x-y}}{(2 - (-1) - 1)(2 - (-1) - 2)}$$

$$= \frac{e^{2x-y}}{2}$$

Hence the solution is $z = e^{x} f_1 (y+x) + e^{2x} f_2(y+x) + \frac{e^{2x-y}}{2}$.

Example 32

Solve $(D^2 – DD' + D' – 1) z = \cos (x + 2y)$

The given equation can be rewritten as

$(D-D'+1) (D-1) z = \cos (x + 2y)$

Here $m_1 = 1$, $m_2 = 0$, $c_1 = -1$, $c_2 = 1$.

Therefore, the C.F $= e^{-x} f_1(y+x) + e^{x} f_2 (y)$

$$\text{P.I} = \frac{1}{(D^2 – DD' + D' – 1)} \cos (x+2y) \quad [\text{Put } D^2 = – 1, DD' = - 2, D'^2 = – 4]$$

$$= \frac{1}{– 1 – (– 2) + D' – 1} \cos (x+2y)$$

$$= \frac{1}{D'} \cos (x+2y)$$

$$= \frac{\sin (x+2y)}{2}$$

Hence the solution is $z = e^{-x} f_1(y+x)\ e^{x} f_2(y) + \frac{\sin(x+2y)}{2}$.

Example 33

Solve $[(D + D'– 1) (D + 2D' – 3)] z = e^{x+2y} + 4 + 3x +6y$

Here $m_1 = – 1$, $m_2 = – 2$, $c_1 = 1$, $c_2 = 3$.

Hence the C.F is $z = e^{x} f_1(y - x) + e^{3x} f_2(y - 2x)$.

$$P.I_1 = \frac{e^{x+2y}}{(D+D' - 1)(D + 2D' - 3)} \quad [\text{Put } D = 1, D' = 2]$$

$$= \frac{e^{x+2y}}{(1+2-1)(1+4-3)}$$

$$= \frac{e^{x+2y}}{4}$$

$$P.I_2 = \frac{1}{(D+D' - 1)(D + 2D' - 3)}(4 + 3x + 6y)$$

$$= \frac{1}{3\,[1 - (D+D')]\left[1 - \frac{D + 2D'}{3}\right]}(4 + 3x + 6y)$$

$$= \frac{1}{3}[1 - (D + D')]^{-1}\left[1 - \frac{D + 2D'}{3}\right]^{-1}(4 + 3x + 6y)$$

$$= \frac{1}{3}[1 + (D + D') + (D+D')^2 + \ldots]\left[1 + \frac{D + 2D'}{3} + \frac{1}{9}(D+2D')^2 + \ldots\ldots\right] . (4 + 3x + 6y)$$

$$= \frac{1}{3}\left[1 + \frac{4}{3}D + \frac{5}{3}D' + \ldots\ldots\ldots\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left(4 + 3x + 6y + \frac{4}{3}(3) + \frac{5}{3}(6)\right.$$

$$= x + 2y + 6$$

Hence the complete solution is

$$z = e^x f_1(y-x) + e^{3x} f_2(y-2x) + \frac{e^{x+2y}}{4} + x + 2y + 6.$$

Exercises

(a) Solve the following homogeneous Equations.

1. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x + y)$

2. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x.\cos 2y$

3. $(D^2 + 3DD' + 2D'^2)\, z = x + y$

4. $(D^2 - DD' + 2D'^2)\, z = xy + e^x.\cosh y$

Hint: $e^x.\cosh y = e^x.\frac{e^y + e^{-y}}{2} = \frac{e^{x+y} + e^{x-y}}{2}$

5. $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(x+2y) + e^{2x+y}$

6. $(D^2 + 4DD' - 5D'^2)\, z = 3e^{2x-y} + \sin(x - 2y)$

7. $(D^2 - DD' - 30D'^2)\, z = xy + e^{6x+y}$

8. $(D^2 - 4D'^2)\, z = \cos 2x.\cos 3y$

9. $(D^2 - DD' - 2D'^2)\, z = (y - 1)e^x$

10. $4r + 12s + 9t = e^{3x - 2y}$

(b) Solve the following non – homogeneous equations.

1. $(2DD' + D'^{2} - 3D')\, z = 3\cos(3x - 2y)$

2. $(D^2 + DD' + D' - 1)\, z = e^{-x}$

3. $r - s + p = x^2 + y^2$

4. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$

5. $(D^2 - D'^2 - 3D + 3D')\, z = xy + 7.$

UNIT–II

FOURIER SERIES

INTRODUCTION

The concept of Fourier series was first introduced by Jacques Fourier (1768–1830), French Physicist and Mathematician. These series became a most important tool in Mathematical physics and had deep influence on the further development of mathematics it self.Fourier series are series of cosines and sines and arise in representing general periodic functions that occurs in many Science and Engineering problems. Since the periodic functions are often complicated, it is necessary to express these in terms of the simple periodic functions of sine and cosine. They play an important role in solving ordinary and partial differential equations.

PERIODIC FUNCTIONS

A function f (x) is called periodic if it is defined for all real „x‟ and if there is some positive number „p‟ such that

$f(x + p) = f(x)$ for all x.

This number „p‟ is called a period of f(x).

If a periodic function f (x) has a smallest period p (>0), this is often called the fundamental period of f(x). For example, the functions cosx and sinx have fundamental period 2π.

DIRICHLET CONDITIONS

Any function f(x), defined in the interval $c \le x \le c + 2\pi$, can be developed as a Fourier series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ provided the following conditions are satisfied.

f (x) is periodic, single– valued and finite in $[c, c + 2\pi]$.
f (x) has a finite number of discontinuities in $[c, c + 2\pi]$.
f (x) has at the most a finite number of maxima and minima in $[c, c + 2\pi]$.

These conditions are known as Dirichlet conditions. When these conditions are satisfied, the Fourier series converges to f(x) at every point of continuity. At a point of discontinuity x = c, the sum of the series is given by

$$f(x) = (1/2) [f(c-0) + f(c+0)] ,$$

where f (c-0) is the limit on the left and f (c+0) is the limit on the right.

EULER'S FORMULAE

The Fourier series for the function f(x) in the interval $c < x < c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \text{ where}$$

$$a_0 = \frac{1}{\pi} \int_{C}^{C+2\pi} f(x)\, dx.$$

$$a_n = \frac{1}{\pi} \int_{C}^{C+2\pi} f(x) \cos nx\, dx.$$

$$b_n = \frac{1}{\pi} \int_{C}^{C+2\pi} f(x) \sin nx\, dx.$$

These values of a_0, a_n, b_n are known as Euler‟s formulae. The coefficients a_0, a_n, b_n are also termed as Fourier coefficients.

Example 1

Expand f(x) = x as Fourier Series (Fs) in the interval [-π, π]

Let $f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$ ----------- (1)

Here $a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ \frac{\pi^2}{2} - \frac{\pi^2}{2} \right\} = 0$$

$a_o = 0$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \, d \left(\frac{\sin nx}{n} \right)$$

$$= \frac{1}{\pi} \left\{ (x) \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ \frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2} \right\}$$

$$= 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \, d \left(\frac{-\cos nx}{n} \right)$$

$$= \frac{1}{\pi} \left\{ (x) \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ -\frac{\pi \cos n\pi}{n} - \frac{\pi \cos n\pi}{n} \right\}$$

$$= \frac{-2\pi \cos n\pi}{n\pi}$$

$$b_n = \frac{2}{n} (-1)^{n+1} \qquad [\because \cos n\pi = (-1)^n]$$

Substituting the values of a_o, a_n & b_n in equation (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$x = 2 \left\{ \frac{\sin x}{1} - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \ldots \right.$$

Example 2

Expand $f(x) = x^2$ as a Fourier Series in the interval $(-\pi \le x \le \pi)$ and hence deduce that

1. $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\dots\dots\dots = \frac{\pi^2}{12}$

2. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots\dots\dots = \frac{\pi^2}{6}$

3. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\dots\dots\dots = \frac{\pi^2}{8}$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

Here

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx$$

$$= \frac{1}{\pi} \left\{ \frac{x^3}{3} \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left(\frac{\pi^3}{3} + \frac{\pi^3}{3} \right)$$

$$a_o = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, d \left(\frac{\sin nx}{n} \right)$$

$$= \frac{1}{\pi} \left\{ \left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\frac{-\cos nx}{n^2} + (2) - \frac{\sin nx}{n^3} \right] \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left(\frac{2\pi \cos n\pi}{n^2} + \frac{2\pi \cos n\pi}{n^2} \right)$$

$$a_n = \frac{4}{n^2} (-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, d \left(\frac{-\cos nx}{n} \right)$$

$$= \frac{1}{\pi} \left\{ \left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\frac{-\sin nx}{n^2} + (2)\frac{\cos nx}{n^3} \right] \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ \frac{-\pi^2 \cos n\pi}{n} + \frac{\pi^2 \cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - \frac{2\cos n\pi}{n^3} \right\}$$

$$b_n = 0$$

Substituting the values of a_0, a_n & b_n in equation (1) we get

$$f(x) = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$

i.e, $$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$

i.e, $$x^2 = \pi^2 + 4 \sum^{\infty} (-1)^n \cos nx$$

$$\frac{\ }{3} \quad \sum_{n=1} \frac{\ }{n^2}$$

$$= \frac{\pi^2}{3} + 4 \left(\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots \right)$$

$$\therefore x^2 = \frac{\pi^2}{3} - 4 \left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots \right) \text{----------------(2)}$$

Put x = 0 in equation (2) we get

$$0 = \frac{\pi^2}{3} - 4 \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \right)$$

$$\text{i.e,} \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots = \frac{\pi^2}{12} \text{-----------------------------} (3)$$

Put x = π in equation (2) we get

$$\pi^2 = \frac{\pi^2}{3} - 4 \left(\frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \ldots \right)$$

$$\text{i.e,} \quad \pi^2 - \frac{\pi^2}{3} = 4 \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \right)$$

$$\text{i.e,} \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6} \text{-----------------}(4)$$

Adding equations (3) & (4) we get

$$\left\{ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots \right\} + \left\{ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \right\} = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

$$\text{i.e,} \quad 2 \left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right\} = \frac{3\pi^2}{12}$$

$$\text{i.e,} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \ldots = \frac{\pi^2}{8}$$

Example 3

Obtain the Fourier Series of periodicity 2π for $f(x) = e^x$ in [-π, π]

$$\text{Let} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{-------------(1)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\, dx$$

$$= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$= \frac{2}{2\pi}\{e^{\pi} - e^{-\pi}\}$$

$$a_0 = \frac{2}{\pi}\sin h\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\, dx$$

$$= \frac{1}{\pi}\left\{\frac{e^x}{(1+n^2)}\left[\cos nx + n\sin nx\right]\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{e^{\pi}}{1+n^2}(-1)^n - \left(\frac{e^{-\pi}(-1)^n}{1+n^2}\right.\right.$$

$$= \frac{(-1)^n}{(1+n^2)\pi}\left(e^{\pi} - e^{-\pi}\right)$$

$$a_n = \frac{2(-1)^n}{\pi(1+n^2)}\sin h\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx$$

$$= \frac{1}{\pi}\left\{\frac{e^x}{(1+n^2)}(\sin nx - n\cos nx)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{e^{\pi}\{-n(-1)^n\}}{1+n^2} - \frac{e^{-\pi}\{-n(-1)^n\}}{1+n^2}\right\}$$

$$= \frac{n(-1)^{n+1}}{\pi(1+n^2)}\,(e^{\pi} - e^{-\pi})$$

$$b_n = \frac{2n(-1)^{n+1}\sin h\pi}{\pi(1+n^2)}$$

$$f(x) = \frac{1}{\pi}\sin h\pi + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{\pi(1+n^2)}\sinh\pi\cos nx + \frac{2(-n)(-1)^n}{\pi(1+n^2)}\sinh\pi\sin nx\right.$$

$$e^x = \frac{1}{\pi}\sin h\pi + \frac{2\sin h\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)$$

$$\text{ie, } e^x = \frac{\sin h\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)\right.$$

Example 4

$$\text{Let } f(x) = \begin{cases} x & \text{in } (O, \pi) \\ (2\pi - x) & \text{in } (\pi, 2\pi) \end{cases}$$

Find the FS for f (x) and hence deduce that $\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \text{ -------------------------------(1)}$$

$$\text{Here } a_0 = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[\frac{-(2\pi - x)^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \frac{\pi^2}{2}\right\} = \pi$$

i.e, $a_o = \pi$

$$a_n = \frac{1}{\pi}\left\{\int_o^{\pi} x \cos nx\, dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\, d\left(\frac{\sin nx}{n}\right) + \int_{\pi}^{2\pi} (2\pi - x)\, d\left(\frac{\sin nx}{n}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\left[(x)\frac{\sin nx}{n} - (1)\frac{-\cos nx}{n^2}\right]_o^{\pi} + \left[(2\pi\text{-}x)\frac{\sin nx}{n} - (-1)\frac{-\cos nx}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\cos n\pi}{n^2} - \frac{1}{n^2} - \frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2\cos n\pi}{n^2} - \frac{2}{n^2}\right\}$$

$$a_n = \frac{2}{n^2\pi}\,[(-1)^n - 1]$$

$$bn = \frac{1}{\pi}\left\{\int_o^{\pi} f(x) \sin nx\, dx + \int_{\pi}^{2\pi} f(x) \sin nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_o^{\pi} x\, d\left(\frac{-\cos nx}{n}\right) + \int_{\pi}^{2\pi} (2\pi - x)\, d\left(\frac{-\cos nx}{n}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\left[(x)\frac{-\cos nx}{n} - (1)\frac{-\sin nx}{n}\right]^{\pi} + \left[(2\pi\text{-}x)\frac{-\cos nx}{n} - (-1)\frac{-\sin nx}{n}\right]^{2\pi}\right\}$$

$\pi \quad n \quad n^2 \quad {}_o \quad n \quad n^2 \quad {}_\pi$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{\pi\cos n\pi}{n}\right] = 0$$

i.e, $b_n = 0$.

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right] \qquad \text{-----(2)}$$

Putting x = 0 in equation(2), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

$$\text{i.e,} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$$

$$\text{i.e,} \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Example 5

Find the Fourier series for $f(x) = (x + x^2)$ in $(-\pi < x < \pi)$ of percodicity 2π and hence deduce that $\sum_{n=1}^{\infty} (1/n^2) = \pi^2/6$.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{Here,} \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{2}\left\{\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^{\pi}\right.$$

□

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right.$$

$$a_o = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2) \, d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{1}{\pi}\left\{(x + x^2)\frac{\sin nx}{n} - (1+2x)\frac{-\cos nx}{n^2} + (2)\frac{-\sin nx}{n^3}\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{(1+2\pi)\frac{(-1)^n}{n^2} - (1-2\pi)\frac{(-1)^n}{n^2}\right.$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2) \, d\left(\frac{-\cos nx}{n}\right)$$

$$= \frac{1}{\pi}\left\{(x + x^2)\frac{-\cos nx}{n} - (1+2x)\frac{-\sin nx}{n^2} + (2)\frac{\cos nx}{n^3}\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi^2(-1)^n}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n} + \frac{\pi^2(-1)^n}{n^2}\right.$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right]$$

$$= \frac{\pi^2}{3} - 4\left\{\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right. + 2\left\{\sin x - \frac{\sin 2x}{2} + \ldots\right.$$

Here x = -π and x = π are the end points of the range. ∴ The value of FS at x = π is the average of the values of f(x) at x = π and x = -π.

$$\therefore f(x) = \frac{f(-\pi) + f(\pi)}{2}$$

$$= \frac{-\pi + \pi^2 + \pi + \pi^2}{2}$$

$$= \pi^2$$

Putting x = π, we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right.$$

$$\text{i.e, } \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

$$\text{Hence,} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercises:

Determine the Fourier expressions of the following functions in the given interval

1.$f(x) = (\pi - x)^2$, $0 \leq x \leq 2\pi$

2.$f(x) = 0$ in $-\pi < x < 0$

$= \pi$ in $0 < x < \pi$

3.$f(x) = x - x^2$ in $[-\pi,\pi]$

4.$f(x) = x(2\pi-x)$ in $(0,2\pi)$

5.$f(x) = \sinh ax$ in $[-\pi, \pi]$

6.$f(x) = \cosh ax$ in $[-\pi, \pi]$

7.$f(x) = 1$ in $0 < x < \pi$

$= 2$ in $\pi < x < 2\pi$

8.$f(x) = -\pi/4$ when $-\pi < x < 0$

$= \pi/4$ when $0 < x < \pi$

9.$f(x) = \cos\alpha x$, in $-\pi < x < \pi$, where „α‟ is not an integer

10.Obtain a fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive the series for $\pi / \sinh\pi$

Even and Odd functions

A function f(x) is said to be even if $f(-x) = f(x)$. For example x^2, cosx, x sinx, secx are even functions. A function f (x) is said to be odd if $f(-x) = -f(x)$. For example, x^3, sin x, x cos x,. are odd functions.

(1) The Euler‟s formula for even function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where } a_o = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \; ; \; a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

(2) The Euler‟s formula for odd function is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \dfrac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

Example 6

Find the Fourier Series for f (x) = x in (- π, π)
Here, f(x) = x is an odd function.

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{ ---------------------- (1)}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \, d\left(\frac{-\cos nx}{n}\right)$$

$$= \frac{2}{\pi} \left\{ (x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right) \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left\{ \frac{-\pi \cos n\pi}{n} \right\}$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$\text{i.e,} \quad x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

Example 7

Expand f (x) = |x| in $(-\pi, \pi)$ as FS and hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

Solution

Here f(x) = |x| is an even function.

$$\therefore f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{---} (1)$$

$$a_o = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x\, d\left(\frac{\sin nx}{n} \right)$$

$$= \frac{2}{\pi} \left\{ (x) \frac{\sin nx}{n} - (1) \left(\frac{-\cos nx}{n^2} \right) \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left\{ \frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right\}$$

$$a_n = \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [(-1)^n - 1] \cos nx$$

π 4 cos x cos3x cos5x

i.e, $|x| = \frac{\quad}{2} - \frac{\quad}{\pi} \frac{\quad}{1^2} + \frac{\quad}{3^2} + \frac{\quad}{5^2} + \ldots\ldots$ --------------(2)

Putting x = 0 in equation (2), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots.$$

$$\text{Hence,} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots. = \frac{\pi^2}{8}$$

Example 8

$$\text{If } f(x) = 1 + \frac{2x}{\pi} \text{ in } (-\pi, 0)$$

$$= 1 \frac{2x}{\pi} \text{ in } (0, \pi)$$

Then find the FS for f(x) and hence show that $\sum_{n=1}^{\infty} (2n-1)^{-2} = \pi^2/8$

Here f (-x) in (-π,0) = f (x) in (0,π)

f (-x) in (0,π) = f (x) in (-π,0)

∴ f(x) is a even function

$$\text{Let } f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ --(1).}$$

$$a_o = \frac{2}{\pi} \int_0^{\pi} \left(1 \frac{2x}{\pi} \right) dx$$

$$= \frac{2}{\pi} \left\{ x - \frac{2x^2}{2\pi} \right\}_0^{\pi}$$

$a_0 = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{4}{\pi^2 n^2}[(1-(-1)^n]$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}[1-(-1)^n]\cos nx$$

$$= \frac{4}{\pi^2}\left\{\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} + \dots\dots \right\} \text{-----------(2)}$$

Put x = 0 in equation (2) we get

$$\frac{\pi^2}{4} = 2\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right\}$$

$$\Longrightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

or $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Example 9

Obtain the FS expansion of $f(x) = x \sin x$ in $(-\pi < x < \pi)$ and hence deduce that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\dots = \frac{\pi - 2}{4}.$$

Here f (x) = xsinx is an even function.

$$\text{Let } f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ ----------------------- (1)}$$

$$\text{Now, } a_o = \frac{2}{\pi}\int_0^{\pi} x\sin x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \, d(-\cos x)$$

$$= \frac{2}{\pi}\Big\{ (x)(-\cos x) - (1)(-\sin x) \Big\}_0^{\pi}$$

$$a_0 = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \, [\sin(1+n)x + \sin(1-n)x] \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \, d\left\{ \frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n} \right\}$$

$$= \frac{1}{\pi}\left\{ (x)\left\{ \frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n} \right\} - (1)\left\{ \frac{-\sin(1+n)x}{(1+n)^2} - \frac{\sin(1-n)x}{(1-n)^2} \right\} \right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{ \frac{-\pi\cos(1+n)\pi}{1+n} - \frac{\pi\cos(1-n)\pi}{1-n} \right\}$$

$$- [\cos\pi \cos n\pi - \sin\pi \sin n\pi] \qquad [\cos\pi \cos n\pi - \sin\pi \sin n\pi]$$

$$= \frac{\text{---}}{1+n} - \frac{\text{---}}{1-n}$$

$$= \frac{(1+n)(-1)^n + (1-n)(-1)^n}{1-n^2}$$

$$a_n = \frac{2(-1)^n}{1-n^2} \quad \text{, Provided } n \neq 1$$

When n = 1

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \, d\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{1}{\pi}\left\{(x)\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right\}_0^{\pi}$$

Therefore, $a_1 = -1/2$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2} \cos nx$$

$$\text{ie, } x \sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{3} - \frac{\cos 3x}{8} + \frac{\cos 4x}{15} - \ldots\right\}$$

Putting $x = \pi/2$ in the above equation, we get

$\pi \quad 1 \quad 1 \quad 1$

$$\frac{\quad}{2} - 1 = 2\left(\frac{\quad}{1.3} - \frac{\quad}{3.5} + \frac{\quad}{5.7} - \ldots\ldots\right)$$

Hence, $$\frac{1}{1.3} - \frac{1}{1.5} + \frac{1}{5.7} - \ldots\ldots = \frac{\pi - 2}{4}$$

Exercises:

Determine Fourier expressions of the following functions in the given interval:

i. $f(x) = \pi/2 + x, \ -\pi \leq x \leq 0$
$\quad \pi/2 - x, \ 0 \leq x \leq \pi$

ii. $f(x) = -x+1$ for $+ -\pi \leq x \leq 0$
$\quad x+1$ for $0 \leq x \leq \pi$

iii. $f(x) = |\sin x|, \ -\pi \leq x \leq \pi$

iv. $f(x) = x^3$ in $-\pi \leq x \leq \pi$

v. $f(x) = x\cos x, \ -\pi < x < \pi$

vi. $f(x) = |\cos x|, \ -\pi < x < \pi$

vii. Show that for $-\pi < x < \pi$, $$\sin ax = \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2 - \alpha^2} - \frac{2\sin 2x}{2^2 - \alpha^2} + \frac{3\sin 3x}{3^2 - \alpha^2} - \ldots\ldots\right)$$

HALF RANGE SERIES

It is often necessary to obtain a Fourier expansion of a function for the range $(0, \pi)$ which is half the period of the Fourier series, the Fourier expansion of such a function consists a cosine or sine terms only.

(i) Half Range Cosine Series

The Fourier cosine series for f(x) in the interval $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n x$$

2 $\qquad \pi$

where $a_0 = \dfrac{2}{\pi} \int_0^{\pi} f(x)\, dx$ and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

(ii) Half Range Sine Series

The Fourier sine series for f(x) in the interval $(0,\pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \dfrac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

Example 10

If c is the constant in ($0 < x < \pi$) then show that

$$c = (4c / \pi) \; \{ \sin x + (\sin 3x /3) + \sin 5x / 5) + \ldots\ldots\ldots\ldots \}$$

Given $f(x) = c$ in $(0,\pi)$.

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \rightarrow (1)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} c \sin nx \, dx$$

$$= \frac{2c}{\pi} \left[\frac{-\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{2c}{\pi} \left[\frac{-(-1)^n}{n} + \frac{1}{n} \right]$$

$$b_n = (2c/n\pi) \; [1 - (-1)^n]$$

∞

$$\therefore\ f(x) = \sum_{n=1} (2c / n\pi)\,(1-(-1)^n)\ \sin nx$$

$$\text{i.e, } c = \frac{4c}{\pi}\left\{ \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \dots \dots \right.$$

Example 11

Find the Fourier Half Range Sine Series and Cosine Series for f(x) = x in the interval (0,π).

Sine Series

$$\text{Let}\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{---------------(1)}$$

$$\text{Here}\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\, d\left(-\cos nx / n\right)$$

$$= \frac{2}{\pi}\left\{ \left[(x)\frac{-\cos nx}{n} - (1)\frac{-\sin nx}{n^2} \right] \right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \frac{-\pi(-1)^n}{n} \right.$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

Cosine Series

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{---------------(2)}$$

$$\text{Here}\quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\,d(\sin nx / n)$$

$$= \frac{2}{\pi}\left\{(x)\frac{\sin nx}{n} - (1)\frac{-\cos nx}{n^2}\right\}_0^{\pi}$$

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos nx$$

$$\Rightarrow x = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \ldots\ldots \ldots\right]$$

Example 12

Find the sine and cosine half-range series for the function function .

$f(x) = x , \quad 0 < x \le \pi/2$

$= \pi - x, \; \pi/2 \le x < \pi$

Sine series

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = (2/\pi) \int_0^{\pi} f(x) \sin nx \, dx$$

$$= (2/\pi) \left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right]$$

$$= (2/\pi) \left[\int_0^{\pi/2} x \, d\left(\frac{-\cos nx}{n}\right) + \int_{\pi/2}^{\pi} (\pi - x) \, d\left(\frac{-\cos nx}{n}\right) \right]$$

$$= (2/\pi) \left\{ \left[x \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi/2} + \left[(\pi - x) - \frac{\cos nx}{n} - (-1) - \frac{\sin nx}{n^2} \right]_{\pi/2}^{\pi} \right\}$$

$$= (2/\pi) \left\{ \left(\frac{-(\pi/2)\cos n(\pi/2)}{n} + \frac{\sin n(\pi/2)}{n^2} \right) - \left(\frac{(\pi/2)\cos n(\pi/2)}{n} - \frac{\sin (\pi/2)}{n^2} \right) \right\}$$

$$= (2/\pi) \left\{ \frac{2 \sin n(\pi/2)}{n^2} \right\}$$

$$= \frac{4}{n^2 \pi} \sin (n\pi/2)$$

Therefore, $f(x) = (4/\pi) \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2} \sin nx$

ie, $f(x) = (4/\pi) \left(\sin x - \frac{\sin 3x}{} + \frac{\sin 5x}{} - \text{--------} \right)$

$3^2 \qquad 5^2$

Cosine series

.Let $f(x) = (a_o/2) + \sum_{n=1}^{\infty} a_n \cos nx$., where

$$a_o = (2/\pi)\int_0^{\pi} f(x)\,dx$$

$$=(2/\pi)\left\{\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi-x)\,dx\right\}$$

$$=(2/\pi)\left\{(x^2/2)\Big|_0^{\pi/2} + (\pi x - x^2/2)\Big|_{\pi/2}^{\pi}\right\} = \pi/2$$

$$a_n = (2/\pi)\int_0^{\pi} f(x)\cos nx\,dx$$

$$=(2/\pi)\left\{\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi-x)\cos nx\,dx\right\}$$

$$=(2/\pi)\left\{\int_0^{\pi/2} x\,d\left(\frac{\sin nx}{n}\right) + \int_{\pi/2}^{\pi} (\pi-x)\,d\left(\frac{\sin nx}{n}\right)\right\}$$

$$= (2/\pi)\left\{\left\{x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi/2}\right.$$

$$\left. + (\pi-x)\left(\frac{\sin nx}{n} - (-1) - \frac{\cos nx}{n^2}\right)_{\pi/2}^{\pi}\right\}$$

$$= (2/\pi)\left\{\left(\frac{(\pi/2)\sin n(\pi/2)}{} + \frac{\cos n(\pi/2)}{} - \frac{1}{}\right)\right.$$

$$n \qquad\qquad n^2 \qquad\qquad n^2$$

$$+ - \frac{\cos nx}{n^2} - \frac{(\pi/2)\sin n(\pi/2)}{n} + \frac{\cos n(\pi/2)}{n^2} \Bigg\}\Bigg\}$$

$$=(2/\pi)\left\{\frac{2\cos n(\pi/2) - \{1+(-1)^n\}}{n^2}\right\}$$

$$\text{Therefore, } f(x)= (\pi/4)+(2/\pi)\sum_{n=1}^{\infty} \frac{2\cos n(\pi/2)- \{1+(-1)^n\}}{n^2}\cos nx .$$

$$= (\pi/4)-(2/\pi)\ \cos 2x+ \frac{\cos 6x}{3^2} +\text{-------------}$$

Exercises

1. Obtain cosine and sine series for $f(x) = x$ in the interval $0< x < \pi$. Hence show that $1/1^2 + 1/3^2 + 1/5^2 + \ldots = \pi^2/8$.

2. Find the half range cosine and sine series for $f(x) = x^2$ in the range $0 \leq x \leq \pi$

3. Obtain the half-range cosine series for the function $f(x) = x\sin x$ in $(0,\pi)$..

4. Obtain cosine and sine series for $f(x) = x (\pi-x)$ in $0< x < \pi$

5. Find the half-range cosine series for the function

6. $f(x) = (\pi x) / 4$, $0<x< (\pi/2)$

   $= (\pi/4)(\pi-x)$, $\pi/2 < x < \pi$.

7. Find half range sine series and cosine series for

   $f(x) = x$ in $0<x< (\pi/2)$

   $= 0$ in $\pi/2 < x < \pi$.

8. Find half range sine series and cosine series for the function $f(x) == \pi - x$ in the interval $0 < x < \pi$.
9. Find the half range sine series of $f(x) = x \cos x$ in $(0,\pi)$

10. Obtain cosine series for

$f(x) = \cos x, \quad 0<x< (\pi/2)$

$= 0, \pi/2 < x < \pi.$

Parseval’s Theorem

Root Mean square value of the function f(x) over an interval (a, b) is defined as

$$[f(x)]_{rms} = \frac{\int_a^b [f(x)]^2 \, dx}{b-a}$$

The use of r.m.s value of a periodic function is frequently made in the theory of mechanical vibrations and in electric circuit theory. The r.m.s value is also known as the effective value of the function.

Parseval’s Theorem

If f(x) defined in the interval (c, c+2π), then the Parseval‟s Identity is given by

$$\int_c^{c+2\pi} [f(x)]^2 \, dx = (\text{Range}) \left(\frac{ao^2}{4} + \frac{1}{2} \sum (a_n^2 + b_n^2) \right)$$

$$= (2\pi) \left(\frac{ao^2}{4} + \frac{1}{2} \sum (a_n^2 + b_n^2) \right.$$

Example 13

Obtain the Fourier series for $f(x) = x^2$ in $-\pi < x < \pi$

Hence show that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$

we have $a_o = \frac{2\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$, $b_n = 0$, for all n (Refer Example 2).

By Parseval‟s Theorem, we have

$\pi \qquad ao^2 \qquad \infty$

$$\int_{-\pi} [f(x)]^2 \, dx = 2\pi \frac{\quad}{4} + \frac{1}{2} \sum_{n=1} (a_n^2 + b_n^2)$$

i.e,
$$\int_{-\pi}^{\pi} x^4 \, dx = 2\pi \left(\frac{4\pi^4}{36} + 1/2 \sum_{n=1}^{\infty} \frac{16(-1)^{2n}}{n^4} \right)$$

i.e,
$$\left. \frac{x^5}{5} \right]_{-\pi}^{\pi} = 2\pi \left(\frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4} \right)$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Hence
$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$$

CHANGE OF INTERVAL

In most of the Engineering applications, we require an expansion of a given function over an interval 2ℓ other than 2π.

Suppose f(x) is a function defined in the interval $c < x < c+2\ell$. The Fourier expansion for f(x) in the interval $c<x<c+2\ell$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

where
$$a_0 = \frac{1}{\ell} \int_c^{c+2\ell} f(x)dx$$

$$a_n = \frac{1}{\ell} \int_c^{c+2\ell} f(x) \cos (n\pi x / \ell) \, dx \quad \&$$

$$b_n = \frac{1}{\ell} \int_c^{c+2\ell} f(x) \sin (n\pi x / \ell) \, dx$$

Even and Odd Function

If f(x) is an even function and is defined in the interval (c, c+2 ℓ), then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

where $a_0 = \frac{2}{\ell} \int_0^{\ell} f(x)dx$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos (n\pi x / \ell)\, dx$$

If f(x) is an odd function and is defined in the interval (c, c+2 ℓ), then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin (n\pi x / \ell)\, dx$$

Half Range Series

Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin (n\pi x / \ell)\, dx$$

Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$\text{where } a_0 = \frac{2}{\ell} \int_0^{\ell} f(x)dx$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos(n\pi x / \ell)\, dx$$

Example 14

Find the Fourier series expansion for the function

$$f(x) = (c/\ell)x \quad \text{in } 0<x<\ell$$
$$= (c/\ell)(2\ell - x) \text{ in } \ell<x<2\ell$$

$$\text{Let} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}$$

$$\text{Now, } a_0 = \frac{1}{\ell} \int_0^{2\ell} f(x)dx$$

$$= \frac{1}{\ell} \left[(c/\ell) \int_0^{\ell} x\, dx + (c/\ell) \int_{\ell}^{2\ell} (2\ell - x)\, dx \right]$$

$$= \frac{1}{\ell} \left\{ (c/\ell) \left(x^2 / 2\right)_0^{\ell} + (c/\ell) \left(2\ell x - x^2/2\right)_{\ell}^{2\ell} \right\}$$

$$= \frac{c}{\ell^2} \ell^2 = c$$

$$a_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \cos(n\pi x / \ell)\, dx$$

$$= \frac{1}{\ell} \left[\int_0^{\ell} (c/\ell)x \cos\frac{n\pi x}{\ell} dx + \int_{\ell}^{2\ell} (c/\ell)(2\ell - x) \cos\frac{n\pi x}{\ell} dx \right]$$

$$= \frac{c}{\ell^2} \left[\int_0^{\ell} x\, d\left(\frac{\sin(n\pi x/\ell)}{n\pi/\ell}\right) + \int_{\ell}^{2\ell} (2\ell - x)\, d\left(\frac{\sin(n\pi x/\ell)}{n\pi/\ell}\right) \right]$$

$$= \frac{c}{\ell^2}\left[\left\{(x)\left\{\frac{\sin\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right\} - (1)\left\{\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right\}\right\}_0^{\ell}\right.$$

$$\left. + (2\ell - x)\left\{\frac{\sin\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right\} - (-1)\left\{\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right\}\right\}_{\ell}^{2\ell}\Bigg]$$

$$= \frac{c}{\ell^2}\left\{\left\{\frac{\ell^2 \cos n\pi}{n^2\pi^2} - \frac{\ell^2}{n^2\pi^2} + -\frac{\ell^2 \cos 2n\pi}{n^2\pi^2} + \frac{\ell^2 \cos n\pi}{n^2\pi^2}\right\}\right\}$$

$$= \frac{c}{\ell^2}\ \frac{\ell^2}{n^2\pi^2}\ \{ 2 \cos n\pi - 2 \}$$

$$= \frac{2c}{n^2\pi^2}\ \{ (-1)^n - 1\}$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x) \cdot \sin\frac{n\pi x}{\ell}\ dx$$

$$= \frac{1}{\ell}\ \int_0^{\ell} (c/\ell)x\ \sin\frac{n\pi x}{\ell}\ dx + \int_{\ell}^{2\ell} (c/\ell)(2\ell - x) \sin\frac{n\pi x}{\ell}\ dx$$

$$= \frac{c}{\ell^2}\ \int_0^{\ell} x\ d\ \left(-\frac{\cos(n\pi x/\ell)}{n\pi/\ell}\right) + \int_{\ell}^{2\ell} (2\ell - x)\ d\ \left(-\frac{\cos(n\pi x/\ell)}{n\pi/\ell}\right)\Bigg\}\Bigg\}$$

$$= \frac{c}{\ell^2}\left[\left\{(x)\left\{-\frac{\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right\} - (1)\ -\frac{\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right\}_0^{\ell}\right.$$

$$\left. + \left\{(2\ell - x)\ -\frac{\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right\} - (-1)\left\{-\frac{\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right\}\right\}_{\ell}^{2\ell}$$

$$= \frac{c}{\ell^2}\left\{ -\frac{\ell^2 \cos n\pi}{n\pi} + \frac{\ell^2 \cos n\pi}{n\pi}\right\}$$

$= 0.$

$$\text{Therefore, } f(x) = \frac{c}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty} \frac{\{(-1)^n - 1\}}{n^2} \cos(n\pi x/\ell)$$

Example 15

Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$, in $0 < x < 3$.

Here $2\ell = 3$.

$\therefore\ \ell = 3/2$.

$$\text{Let}\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}$$

$$\text{where}\quad a_o = (2/3)\int_0^3 (2x - x^2)\,dx$$

$$= (2/3)\left\{2(x^2/2) - (x^3/3)\,dx\right\}_0^3$$

$$= 0.$$

$$a_n = (2/3)\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= (2/3)\int_0^3 (2x - x^2)\, d\left(\frac{\sin(2n\pi x/3)}{(2n\pi/3)}\right)$$

$$= (2/3)\left\{(2x - x^2)\left(\frac{\sin(2n\pi x/3)}{(2n\pi/3)}\right) - (2-2x)\left(-\frac{\cos(2n\pi x/3)}{(4n^2\pi^2/9)}\right) + (-2)\left(-\frac{\sin(2n\pi x/3)}{(8n^3\pi^3/27)}\right)\right\}_0^3$$

$$= (2/3)\ \{- (9/n^2\pi^2) - (9/2n^2\pi^2)\} \qquad = -9/n^2\pi^2$$

$$b_n = (2/3)\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= (2/3)\int_0^3 (2x - x^2)\, d\left(-\frac{\cos(2n\pi x/3)}{(2n\pi/3)}\right)$$

$$= (2/3)\left\{(2x - x^2)\left(-\frac{\cos(2n\pi x/3)}{(2n\pi/3)}\right) - (2-2x)\left(-\frac{\sin(2n\pi x/3)}{(4n^2\pi^2/9)}\right) + (-2)\left(\frac{\cos(2n\pi x/3)}{(8n^3\pi^3/27)}\right)\right\}_0^3$$

$$= (2/3)\ \{(9/2n\pi) - (27/4n^3\pi^3) + (27/4n^3\pi^3)\}$$

$$= 3/n\pi$$

Therefore, $$f(x) = \sum_{n=1}^{\infty} \left\{- (9/n^2\pi^2)\cos\frac{2n\pi x}{3} + (3/n\pi)\sin\frac{2n\pi x}{3}\right\}$$

Exercises

1. Obtain the Fourier series for $f(x) = \pi x$ in $0 \le x \le 2$.
2. Find the Fourier series to represent x^2 in the interval $(-\ell, \ell)$.
3. Find a Fourier series in (-2, 2), if
   $f(x) = 0, -2 < x < 0$

$= 1, 0 < x < 2.$

4.Obtain the Fourier series for

$f(x) = 1-x$ in $0 \le x \le \ell$

$= 0$ in $\ell \le x \le 2\ell$. Hence deduce that

$1- (1/3) + (1/5) - (1/7) + \dots = \pi/4$ &

$(1/1^2) + (1/3^2) + (1/5^2) + \dots = (\pi^2/8)$

5.If $f(x) = \pi x, \quad 0 \le x \le 1$

$= \pi(2-x), \; 1 \le x \le 2,$

Show that in the interval (0,2),

$$f(x) = (\pi/2) - (4/\pi)\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots\right)$$

6.Obtain the Fourier series for

$f(x) = x$ in $0 < x < 1$

$= 0$ in $1 < x < 2$

7.Obtain the Fourier series for

$f(x) = (cx/\ell)$ in $0 < x < \ell$

$= (c/\ell)(2\ell - x)$ in $\ell < x < 2\ell$.

8.Obtain the Fourier series for

$f(x) = (\ell + x), \; -\ell \le x \le 0.$

$= (\ell - x), \; 0 \le x \le \ell.$

Deduce that $$\sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

9.Obtain half-range sine series for the function

$f(x) = cx \quad$ in $0 < x \le (\ell/2)$

$= c(\ell - x)$ in $(\ell/2) < x < \ell$

10. Express $f(x) = x$ as a half – range sine series in $0 < x < 2$

11.Obtain the half-range sine series for e^x in $0 < x < 1$.

12.Find the half –range cosine series for the function $f(x) = (x-2)^2$ in the interval $0 < x < 2$.

$$\text{Deduce that} \quad \sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Harmonic Analysis

The process of finding the Fourier series for a function given by numerical values is known as harmonic analysis.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \text{ where}$$

$$\text{ie, } f(x) = (a_0/2) + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \ldots \quad \text{---------------(1)}$$

$$\text{Here } a_0 = 2\,[\text{mean values of } f(x)] = \frac{2 \sum f(x)}{n}$$

$$a_n = 2\,[\text{mean values of } f(x)\cos nx] = \frac{2 \sum f(x) \cos nx}{n}$$

$$\& \quad b_n = 2\,[\text{mean values of } f(x)\sin nx] = \frac{2 \sum f(x) \sin nx}{n}$$

In (1), the term $(a_1 \cos x + b_1 \sin x)$ is called the **fundamental or first harmonic**, the term $(a_2 \cos 2x + b_2 \sin 2x)$ is called the **second harmonic** and so on.

Example 16

Compute the first three harmonics of the Fourier series of f(x) given by the following table.

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x):	1.0	1.4	1.9	1.7	1.5	1.2	1.0

We exclude the last point $x = 2\pi$.

Let $f(x) = (a_0/2) + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots\ldots\ldots\ldots$

To evaluate the coefficients, we form the following table.

x	f(x)	cosx	sinx	cos2x	sin2x	cos3x	sin3x
0	1.0	1	0	1	0	1	0
π/3	1.4	0.5	0.866	-0.5	0.866	-1	0
2π/3	1.9	-0.5	0.866	-0.5	-0.866	1	0
π	1.7	-1	0	1	0	-1	0
4π/3	1.5	-0.5	-0.866	-0.5	0.866	1	0
5π/3	1.2	0.5	-0.866	-0.5	-0.866	-1	0

$$\text{Now, } a_0 = \frac{2\sum f(x)}{6} = \frac{2\,(1.0 + 1.4 + 1.9 + 1.7 + 1.5 + 1.2)}{6} = 2.9$$

$$a_1 = \frac{2\sum f(x)\cos x}{6} = -0.37$$

$$a_2 = \frac{2\sum f(x)\cos 2x}{6} = -0.1$$

$$a_3 = \frac{2\sum f(x)\cos 3x}{6} = 0.033$$

$$b_1 = \frac{2\sum f(x)\sin x}{6} = 0.17$$

$$b_2 = \frac{2\sum f(x)\sin 2x}{6} = -0.06$$

$$b_3 = \frac{2\sum f(x)\sin 3x}{6} = 0$$

$\therefore\ f(x) = 1.45 – 0.37\cos x + 0.17 \sin x – 0.1\cos 2x – 0.06 \sin 2x + 0.033 \cos 3x + \ldots$

Example 17

Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:

x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

Taking the interval as 60°, we have

θ:	0°	60°	120°	180°	240°	300°
x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

$\therefore$ Fourier cosine series in the interval (0, 2π) is

$$y = (a_0 /2) + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \ldots..$$

To evaluate the coefficients, we form the following table.

θ^o	cosθ	cos2θ	cos3θ	y	y cosθ	y cos2θ	y cos3θ
0^o	1	1	1	4	4	4	4
60^o	0.5	-0.5	-1	8	4	-4	-8
120^o	-0.5	-0.5	1	15	-7.5	-7.5	15
180^o	-1	1	-1	7	-7	7	-7
240^o	-0.5	-0.5	1	6	-3	-3	6
300^o	0.5	-0.5	-1	2	1	-1	-2
			Total	42	-8.5	-4.5	8

Now, $a_0 = 2\ (42/6) = 14$

$a_1 = 2\ (\ -8.5/6) = -\ 2.8$

$a_2 = 2\ (-4.5/6) = -\ 1.5$

$a_3 = 2\ (8/6) = 2.7$

$\therefore y = 7 - 2.8 \cos\theta - 1.5 \cos 2\theta + 2.7 \cos 3\theta + \ldots..$

Example 18

The values of x and the corresponding values of f(x) over a period T are given below. Show that $f(x) = 0.75 + 0.37 \cos\theta + 1.004 \sin\theta$, where $\theta = (2\pi x)/T$

x:	0	T/6	T/3	T/2	2T/3	5T/6	T
y:	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

We omit the last value since f(x) at x = 0 is known.

Here $\theta = \dfrac{2\pi x}{T}$

When x varies from 0 to T, θ varies from 0 to 2π with an increase of 2π /6.

Let $f(x) = F(\theta) = (a_0/2) + a_1 \cos\theta + b_1 \sin\theta$.

To evaluate the coefficients, we form the following table.

θ	y	cosθ	sinθ	y cosθ	y sinθ
0	1.98	1.0	0	1.98	0
π/3	1.30	0.5	0.866	0.65	1.1258
2π/3	1.05	-0.5	0.866	-0.525	0.9093
Π	1.30	-1	0	-1.3	0
4π/3	-0.88	-0.5	-0.866	0.44	0.762
5π/3	-0.25	0.5	-0.866	-0.125	0.2165
	4.6			1.12	3.013

Now, $a_0 = 2\ (\sum f(x) / 6) = 1.5$

$a_1 = 2\ (1.12\ /6) = \ 0.37$

$a_2 = 2\ (3.013/6) = \ 1.004$

Therefore, $f(x) = 0.75 + 0.37 \cos\theta + 1.004 \sin\theta$

Exercises

1. The following table gives the variations of periodic current over a period.

t (seconds) :	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amplitude):	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a direct current part of 0.75 amp in the variable current and obtain the amplitude of the first harmonic.

2. The turning moment T is given for a series of values of the crank angle θ° = 75°

θ°	:	0	30	60	90	120	150	180
T°	:	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T for θ = 75°

3. Obtain the constant term and the co-efficient of the first sine and cosine terms in the Fourier expansion of „y" as given in the following table.

X	:	0	1	2	3	4	5
Y	:	9	18	24	28	26	20

4. Find the first three harmonics of Fourier series of y = f(x) from the following data.

X : 0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
Y : 298	356	373	337	254	155	80	51	60	93	147	221

Complex Form of Fourier Series

The series for f(x) defined in the interval (c, c+2π) and satisfying Dirichlet"s conditions can be given in the form of $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$,

where ,

$$c_n = \frac{1}{2\pi} \int_c^{c+2\pi} f(x)\, e^{-inx}\, dx$$

In the interval (c, c+2ℓ), the complex form of Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{\ell}}$$

where, $$c_n = \frac{1}{2\ell} \int_c^{c+2\ell} f(x)\, e^{\frac{-in\pi x}{\ell}}\, dx$$

Example 19

Find the complex form of the Fourier series $f(x) = e^{-x}$ in $-1 \le x \le 1$.

We have $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i n \pi x}$$

where $$c_n = \frac{1}{2} \int_{-1}^{1} e^{-x} e^{-i n \pi x} dx$$

$$c_n = \frac{1}{2} \int_{-1}^{1} e^{-(1+i n \pi) x} dx$$

$$= \frac{1}{2} \left(\frac{e^{-(1+i n\pi) x}}{-(1+in\pi)} \right)_{-1}^{1}$$

$$= \frac{1}{-2(1+in\pi)} \left[e^{-(1+i n \pi)} - e^{(1+i n\pi)} \right]$$

$$= \frac{(1-in\pi)}{-2(1+n^2\pi^2)} [e^{-1} (\cos n\pi - i\sin n\pi) - e (\cos n\pi + i \sin n\pi)]$$

$$= \frac{(1-in\pi)}{-2(1+n^2\pi^2)} \cos n\pi (e^{-1} - e)$$

$$C_n = \frac{(1-in\pi)}{(1+n^2\pi^2)} (-1)^n \sinh 1$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(1-in\pi)}{(1+n^2\pi^2)} (-1)^n \sinh 1 \; e^{i n \pi x}$$

Example 20

Find the complex form of the Fourier series $f(x) = e^x$ in $-\pi < x < \pi$.

We have $$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i nx}$$

where $$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x} \; e^{-i\,nx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-i\,n)\,x}\, dx$$

$$= \frac{1}{2\pi} \left(\frac{e^{(1-in)x}}{(1-in)} \right)_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-in)} \left[e^{(1-in)\pi} - e^{-(1-i\,n)\,\pi} \right]$$

$$= \frac{(1+in)}{2\pi(1+n)^2} \left[e^{\pi} (\cos n\pi - i \sin n\pi) - e^{-\pi} (\cos n \pi + i \sin n\pi) \right]$$

$$= \frac{(1+in)}{(1+n^2)} \; (-1)^n \,.\, \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

$$= \frac{(-1)^n (1+in) \sin h\, \pi}{(1+n^2)\, \pi}$$

$$\therefore \quad f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1+in) \sin h\, \pi}{(1+n^2)\, \pi} e^{i\,nx}$$

Exercises

Find the complex form of the Fourier series of the following functions.

1.$f(x) = e^{ax}$, $-\ell < x < \ell$.

2.$f(x) = \cos ax$, $-\pi < x < \pi$.

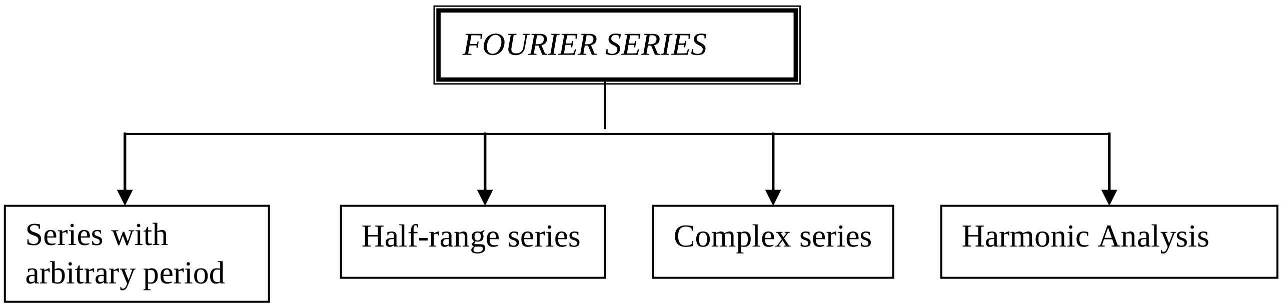
3.$f(x) = \sin x$, $0 < x < \pi$.

4.$f(x) = e^{-x}$, $-1 < x < 1$.

5.$f(x) = \sin ax$, a is not an integer in $(-\pi, \pi)$.

SUMMARY(FOURIER SERIES)

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.



FORMULA FOR FOURIER SERIES

Consider a real-valued function f(x) which obeys the following conditions called Dirichlet‟s conditions :

1. f(x) is defined in an interval (a,a+2*l*), and f(x+2*l*) = f(x) so that f(x) is a periodic function of period 2*l*.
2. f(x) is continuous or has only a finite number of discontinuities in the interval (a,a+2*l*).
3. f(x) has no or only a finite number of maxima or minima in the interval (a,a+2*l*).

Also, let

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)\,dx \qquad (1)$$

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx, \qquad n = 1,2,3,..... \quad (2)$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx, \qquad n = 1,2,3,.\,...... \quad (3)$$

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}\right)x + b_n\sin\left(\frac{n\pi}{l}\right)x \qquad (4)$$

is called the Fourier series of f(x) in the interval (a,a+2*l*). Also, the real numbers a_0, a_1, a_2, ….a_n, and b_1, b_2 , ….b_n are called the Fourier coefficients of f(x). The formulae (1), (2) and (3) are called Euler‟s formulae.

It can be proved that the sum of the series (4) is f(x) if f(x) is continuous at x. Thus we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \ldots\ldots \quad (5)$$

Suppose f(x) is discontinuous at x, then the sum of the series (4) would be

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

where $f(x^+)$ and $f(x^-)$ are the values of f(x) immediately to the right and to the left of f(x) respectively.

Particular Cases

Case (i)

Suppose a=0. Then f(x) is defined over the interval (0,2*l*). Formulae (1), (2), (3) reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,\ldots.\infty \qquad (6)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (0,2*l*).

If we set $l=\pi$, then f(x) is defined over the interval (0,2π). Formulae (6) reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx, \quad n=1,2,\ldots..\infty \qquad (7)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx \qquad n=1,2,\ldots..\infty$$

Also, in this case, (5) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad (8)$$

Case (ii)

Suppose a=-*l*. Then f(x) is defined over the interval (-*l* , *l*). Formulae (1), (2) (3) reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \qquad n = 1,2,\ldots\ldots \infty \qquad (9)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (-*l* , *l*).

If we set *l* = π, then f(x) is defined over the interval (-π, π). Formulae (9) reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \qquad n=1,2,\ldots..\infty \qquad (10)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx \qquad n=1,2,\ldots..\infty$$

Putting *l* = π in (5), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Some useful results :

1. The following rule called Bernoulli's generalized rule of integration by parts is useful in evaluating the Fourier coefficients.

$$\int uvdx = uv_1 - u'v_2 + u''v_3 + \ldots\ldots.$$

Here u', u'' ,….. are the successive derivatives of u and

$$v_1 = \int vdx, v_2 = \int v_1 dx, \ldots\ldots$$

We illustrate the rule, through the following examples :

$$\int x^2 \sin nxdx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$$

$$\int x^3 e^{2x} dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right)$$

2. The following integrals are also useful :

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

3. If „n" is integer, then

$\sin n\pi = 0$, $\cos n\pi = (-1)^n$, $\sin 2n\pi = 0$, $\cos 2n\pi = 1$

ASSIGNMENT

1. The displacement y of a part of a mechanism is tabulated with corresponding angular movement x^0 of the crank. Express y as a Fourier series upto the third harmonic.

x^0	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

2. Obtain the Fourier series of y upto the second harmonic using the following table :

x^0	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	-1.5	0	2.8	3.4

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table :

x	**0**	**1**	**2**	**3**	**4**	**5**
y	9	18	24	28	26	20

4. Find the Fourier series of y upto the second harmonic from the following table :

x	**0**	**2**	**4**	**6**	**8**	**10**	**12**
Y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

5. Obtain the first 3 coefficients in the Fourier cosine series for y, where y is given below

x	**0**	**1**	**2**	**3**	**4**	**5**
y	4	8	15	7	6	2

UNIT – III

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

INTRODUCTION

In Science and Engineering problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with the boundary conditions constitutes a boundary value problem. In the case of ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same method is not applicable to partial differential equations because the general solution contains arbitrary constants or arbitrary functions. Hence it is difficult to adjust these constants and functions so as to satisfy the given boundary conditions. Fortunately, most of the boundary value problems involving linear partial differential equations can be solved by a simple method known as the **method of separation of variables** which furnishes particular solutions of the given differential equation directly and then these solutions can be suitably combined to give the solution of the physical problems.

Solution of the wave equation

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{------------------} (1) .$$

Let y = X(x) . T(t) be the solution of (1), where „X" is a function of „x" only and „T" is a function of „t" only.

Then $\frac{\partial^2 y}{\partial t^2} = X T''$ and $\frac{\partial^2 y}{\partial x^2} = X'' T.$

Substituting these in (1), we get

$$X T'' = a^2 X'' T.$$

i.e, $$\frac{X''}{X} = \frac{T''}{a^2 T} \quad \text{--------------------------} (2).$$

Now the left side of (2) is a function of „x‟ only and the right side is a function of „t‟ only. Since „x‟ and „t‟ are independent variables, (2) can hold good only if each side is equal to a constant.

Therefore, $$\frac{X''}{X} = \frac{T''}{a^2 T} = k \text{ (say)}.$$

Hence, we get $X'' - kX = 0$ and $T'' - a^2 kT = 0$ --------------- (3).

Solving equations (3), we get

(i) when „k‟ is positive and $k = \lambda^2$, say

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$
$$T = c_3 e^{a\lambda t} + c_4 e^{-a\lambda t}$$

(ii) when „k‟ is negative and $k = -\lambda^2$, say

$$X = c_5 \cos\lambda x + c_6 \sin \lambda x$$
$$T = c_7 \cos a\lambda t + c_8 \sin a\lambda t$$

(iii) when „k‟ is zero.

$$X = c_9 x + c_{10}$$
$$T = c_{11} t + c_{12}$$

Thus the various possible solutions of the wave equation are

$$y = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{a\lambda t} + c_4 e^{-a\lambda t}) \text{ ------------------------ } (4)$$
$$y = (c_5 \cos\lambda x + c_6 \sin \lambda x)(c_7 \cos a\lambda t + c_8 \sin a\lambda t) \text{ -------------- } (5)$$
$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \text{ --- } (6)$$

Of these three solutions, we have to select that particular solution which suits the physical nature of the problem and the given boundary conditions. Since we are dealing with problems on vibrations of strings, „y‟ must be a periodic function of „x‟ and „t‟.

Hence the solution must involve trigonometric terms.

Therefore, the solution given by (5),

i. e, $y = (c_5 \cos\lambda x + c_6 \sin \lambda x)(c_7 \cos a\lambda t + c_8 \sin a\lambda t)$

is the only suitable solution of the wave equation.

llustrative Examples.

Example 1

If a string of length ℓ is initially at rest in equilibrium position and each of its points is given

the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_o \sin\frac{\pi x}{\ell}$ **,** $0 < x < \ell$. Determine the displacement y(x,t).

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{ --------------------- (1)}$$

The boundary conditions are

i. $y(0,t) = 0$, for $t \geq 0$.
ii. $y(\ell,t) = 0$, for $t \geq 0$.
iii. $y(x,0) = 0$, for $0 \leq x \leq \ell$.

iv. $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_o \sin\frac{\pi x}{\ell}$ **,** for $0 \leq x \leq \ell$.

Since the vibration of a string is periodic, therefore, the solution of (1) is of the form

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ------------(2)}$$

Using (i) in (2) , we get

$$0 = A(C\cos\lambda at + D\sin\lambda at) \text{ , for all } t \geq 0.$$

Therefore, $A = 0$

Hence equation (2) becomes

$$y(x,t) = B \sin\lambda x(C\cos\lambda at + D\sin\lambda at) \text{------------ } (3)$$

Using (ii) in (3), we get

$0 = B\sin\lambda\ell\ (C\cos\lambda at + D\sin\lambda at)$, for all $t \geq 0$, which gives $\lambda\ell = n\pi$.

Hence, $\lambda = \dfrac{n\pi}{\ell}$, n being an integer.

$$\text{Thus, } y(x,t) = B\sin\frac{n\pi x}{\ell}\left(C\cos\frac{n\pi at}{\ell} + D\sin\frac{n\pi at}{\ell}\right) \text{------------------------ } (4)$$

Using (iii) in (4), we get

$$0 = B\sin\frac{n\pi x}{\ell} . C$$

which implies $C = 0$.

$$\therefore \quad y(x,t) = B\sin\frac{n\pi x}{\ell} . \quad D\sin\frac{n\pi at}{\ell}$$

$$= B_1\sin\frac{n\pi x}{\ell} . \sin\frac{n\pi at}{\ell}, \text{ where } B_1 = BD.$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \sin\frac{n\pi at}{\ell} \text{----------------------------}(5)$$

Differentiating (5) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} . \cos\frac{n\pi at}{\ell} . \frac{n\pi a}{\ell}$$

Using condition (iv) in the above equation, we get

$$v_o \sin\frac{\pi x}{\ell} = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi a}{\ell} \cdot \sin\frac{n\pi x}{\ell}$$

$$\text{i.e,}\quad v_o \sin\frac{\pi x}{\ell} = B_1 \cdot \frac{\pi a}{\ell} \cdot \sin\frac{\pi x}{\ell} + B_2 \cdot \frac{2\pi a}{\ell} \cdot \sin\frac{2\pi x}{\ell} + \cdots$$

Equating like coefficients on both sides, we get

$$B_1 \frac{\pi a}{\ell} = v_o,\quad B_2 \cdot \frac{2\pi a}{\ell} = 0,\quad B_3 \frac{3\pi a}{\ell} = 0, \cdots$$

$$\text{i.e,}\quad B_1 = \frac{v_o \ell}{\pi a},\quad B_2 = B_3 = B_4 = B_5 = \cdots = 0.$$

Substituting these values in (5), we get the required solution.

$$\text{i.e,}\quad y(x,t) = \frac{v_o \ell}{\pi a} \sin\frac{\pi x}{\ell} \cdot \sin\frac{\pi at}{\ell}$$

Example 2

A tightly stretched string with fixed end points x = 0 & x = ℓ is initially at rest in its equilibrium position . If it is set vibrating by giving to each of its points a velocity
$\partial y/\partial t = kx(\ell - x)$ at t = 0. Find the displacement y(x,t).

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{-----------------------} (1)$$

The boundary conditions are

i. $y(0,t) = 0$, for $t \geq 0$.
ii. $y(\ell,t) = 0$, for $t \geq 0$.
iii. $y(x,0) = 0$, for $0 \leq x \leq \ell$.
iv. $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = kx(\ell - x)$, for $0 \leq x \leq \ell$.

Since the vibration of a string is periodic, therefore, the solution of (1) is of the form

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ------------(2)}$$

Using (i) in (2) , we get

$$0 = A(C\cos\lambda at + D\sin\lambda at) \text{, for all } t \geq 0.$$

which gives $A = 0$.

Hence equation (2) becomes

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \text{------------ (3)}$$

Using (ii) in (3), we get

$$0 = B\sin\lambda\ell(C\cos\lambda at + D\sin\lambda at)\text{, for all } t \geq 0.$$

which implies $\lambda\ell = n\pi$.

Hence, $\lambda = \dfrac{n\pi}{\ell}$, n being an integer.

Thus , $y(x,t) = B\sin\dfrac{n\pi x}{\ell}\left(C\cos\dfrac{n\pi at}{\ell} + D\sin\dfrac{n\pi at}{\ell}\right)$ ------------------------------(4)

Using (iii) in (4), we get

$$0 = B\sin\frac{n\pi x}{\ell}.C$$

Therefore, $C = 0$.

Hence, $y(x,t) = B\sin\dfrac{n\pi x}{\ell}.\ D\sin\dfrac{n\pi at}{\ell}$

$$= B_1\sin\frac{n\pi x}{\ell}.\sin\frac{n\pi at}{\ell}\text{, where } B_1 = BD.$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \sin\frac{n\pi at}{\ell} \quad \text{-----------------------(5)}$$

Differentiating (5) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} .\cos\frac{n\pi at}{\ell} . \frac{n\pi a}{\ell}$$

Using (iv), we get

$$kx(\ell-x) = \sum_{n=0}^{\infty} B_n.\frac{n\pi a}{\ell}. \sin\frac{n\pi x}{\ell}$$

$$\text{i.e,} \quad B_n.\frac{n\pi a}{\ell} = \frac{2}{\ell}\int_0^{\ell} f(x).\sin\frac{n\pi x}{\ell}\,dx$$

$$\text{i.e,} \quad B_n = \frac{2}{n\pi a}\int_0^{\ell} f(x).\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{2}{n\pi a}\int_0^{\ell} kx(\ell-x)\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{2k}{n\pi a}\int_0^{\ell} (\ell x - x^2)\, d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi x}{\ell}}\right)$$

$$= \frac{2k}{n\pi a}\quad (\ell x - x^2)\, d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right) - (\ell-2x)\left(\frac{-\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right)$$

$$= \frac{2k}{n\pi a}\left\{\frac{-2\cos n\pi}{\frac{n^3\pi^3}{\ell^3}} + \frac{2}{\frac{n^3\pi^3}{\ell^3}}\right\}$$

$$= \frac{2k}{n\pi a} \cdot \frac{2\ell^3}{n^3\pi^3}\{1 - \cos n\pi\}$$

i.e, $$B_n = \frac{4\,k\ell^3}{n^4\pi^4\,a}\{1 - (-1)^n\}$$

or $$B_n = \begin{cases} \dfrac{8k\ell^3}{n^4\pi^4\,a}, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases}$$

Substituting in (4), we get

$$y(x,t) = \sum_{n=1,3,5,\ldots\ldots}^{\infty} \frac{8k\ell^3}{n^4\pi^4\,a} \sin\frac{n\pi at}{\ell} \sin\frac{n\pi x}{\ell}$$

Therefore the solution is

$$y(x,t) = \frac{8k\ell^3}{\pi^4\,a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi at}{\ell} \sin\frac{(2n-1)\pi x}{\ell}$$

Example 3

A tightly stretched string with fixed end points $x = 0$ & $x = \ell$ is initially in a position given by $y(x,0) = y_0\sin^3(\pi x/\ell)$. If it is released from rest from this position, find the displacement y at any time and at any distance from the end $x = 0$.

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \quad \text{-----------------------} \quad (1)$$

The boundary conditions are

(i) $y(0,t) = 0, \forall\ t \geq 0.$

(ii) $y(\ell,t) = 0, \forall\ t \geq 0.$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, for $0 < x < \ell$.

(iv) $y(x,0) = y_0 \sin^3((\pi x/\ell)$, for $0 < x < \ell$.

The suitable solution of (1) is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ------------(2)}$$

Using (i) and (ii) in (2) , we get

$$A = 0 \quad \& \quad \lambda = \frac{n\pi}{\ell}$$

$$\therefore \quad y(x,t) = B \sin \frac{n\pi x}{\ell} \left(C\cos \frac{n\pi at}{\ell} + D\sin \frac{n\pi at}{\ell}\right) \text{ -----------(3)}$$

$$\text{Now, } \frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{\ell} \left[- C\sin \frac{n\pi at}{\ell} \cdot \frac{n\pi a}{\ell} + D\cos \frac{n\pi at}{\ell} \cdot \frac{n\pi a}{\ell}\right]$$

Using (iii) in the above equation , we get

$$0 = B \sin \frac{n\pi x}{\ell} \quad D \frac{n\pi a}{\ell}$$

Here, B can not be zero . Therefore D = 0.

Hence equation (3) becomes

$$y(x,t) = B \sin \frac{n\pi x}{\ell} \cdot C\cos \frac{n\pi at}{\ell}$$

$$= B_1 \sin \frac{n\pi x}{\ell} \cdot \cos \frac{n\pi at}{\ell}, \text{ where } B_1 = BC$$

The most general solution is

$$\infty \qquad n\pi x \qquad n\pi at$$

$$y(x,t) = \sum_{n=1} B_n \sin \frac{\quad}{\ell} \quad \cos \frac{\quad}{\ell} \text{-------------------------(4)}$$

Using (iv), we get

$$y_0 \sin^3 \frac{n\pi}{\ell} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\ell}$$

$$\text{i.e, } \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\ell} = y_0 \left\{ \frac{3}{4} \sin \frac{\pi x}{\ell} - \frac{1}{4} \sin \frac{3\pi x}{\ell} \right\}$$

$$\text{i.e, } B_1 \sin \frac{\pi x}{\ell} + B_2 \sin \frac{2\pi x}{\ell} + B_3 \sin \frac{3\pi x}{\ell} + \ldots.$$

$$= \frac{3y_0}{4} \sin \frac{\pi x}{\ell} - \frac{y_0}{4} \sin \frac{3\pi x}{\ell}$$

Equating the like coefficients on both sides, we get

$$B_1 = \frac{3y_0}{4}, \; B_3 = \frac{-y_0}{4}, \; B_2 = B_4 = \ldots = 0.$$

Substituting in (4), we get

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{\ell} \cdot \cos \frac{\pi a t}{\ell} - \frac{y_0}{4} \sin \frac{3\pi x}{\ell} \cdot \cos \frac{3\pi a t}{\ell}$$

Example 4

A string is stretched & fastened to two points x = 0 and x = ℓ apart. Motion is started by displacing the string into the form $y(x,0) = k(\ell x - x^2)$ from which it is released at time t = 0. Find the displacement y(x,t).

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{--------------------- (1)}$$

The boundary conditions are

(i) $y(0,t) = 0, \forall\, t \geq 0.$

(ii) $y(\ell,t) = 0, \forall\, t \geq 0.$

(iii) $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \text{ for } 0 < x < \ell.$$

(iv) $y(x,0) = k(\ell x - x^2)$, for $0 < x < \ell$.

The suitable solution of (1) is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ------------(2)}$$

Using (i) and (ii) in (2) , we get

$$A = 0 \quad \& \quad \lambda = \frac{n\pi}{\ell}.$$

$$\therefore \quad y(x,t) = B\sin\frac{n\pi x}{\ell}\left(C\cos\frac{n\pi at}{\ell} + D\sin\frac{n\pi at}{\ell}\right) \text{ -----------(3)}$$

$$\text{Now, } \frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{\ell}\left(-C\sin\frac{n\pi at}{\ell}\cdot\frac{n\pi a}{\ell} + D\cos\frac{n\pi at}{\ell}\cdot\frac{n\pi a}{\ell}\right)$$

Using (iii) in the above equation , we get

$$0 = B\sin\frac{n\pi x}{\ell}\; D\;\frac{n\pi a}{\ell}$$

Here, B can not be zero

$D = 0$

Hence equation (3) becomes

$$y(x,t) = B\sin\frac{n\pi x}{\ell}\cdot C\cos\frac{n\pi at}{\ell}$$

$$= B_1\sin\frac{n\pi x}{\ell}\cdot\cos\frac{n\pi at}{\ell}, \text{ where } B_1 = BC$$

The most general solution is

$$y(x,t) = \sum^{\infty} B_n\sin\frac{n\pi x}{\ell}\cos\frac{n\pi at}{} \text{ ------------------ (4)}$$

n=1 ℓ ℓ

Using (iv), we get $$kx(\ell x - x^2) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \quad \text{------------------------ (5)}$$

The RHS of (5) is the half range Fourier sine series of the LHS function .

$$\therefore B_n = \frac{2}{\ell}\int_0^{\ell} f(x)\,.\,\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{2k}{\ell}\int_0^{\ell} (\ell x - x^2)\, d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right)$$

$$= \frac{2k}{\ell}\left[(\ell x - x^2)\, d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right) - (\ell - 2x)\left(\frac{-\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{\ell}}{\frac{n^3\pi^3}{\ell^3}}\right)\right]_0^{\ell}$$

$$= \frac{2k}{\ell}\left\{\frac{-2\cos n\pi}{\frac{n^3\pi^3}{\ell^3}} + \frac{2}{\frac{n^3\pi^3}{\ell^3}}\right\}$$

$$= \frac{2k}{\ell}\,.\,\frac{2\ell^3}{n^3\pi^3}\{1 - \cos n\pi\}$$

i.e, $$B_n = \frac{4k\ell^2}{n^3\pi^3}\{1 - (-1)^n\}$$

$$\text{or}\quad B_n = \begin{cases} \dfrac{8k\ell^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore y(x,t) = \sum_{n=odd}^{\infty} \frac{8k\ell^2}{n^3\pi^3} \cos\frac{n\pi at}{\ell} . \sin\frac{n\pi x}{\ell}$$

$$\text{or}\qquad y(x,t) = \frac{8k}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos\frac{(2n-1)\pi at}{\ell} . \sin\frac{(2n-1)\pi x}{\ell}$$

Example 5

A uniform elastic string of length 2ℓ is fastened at both ends. The midpoint of the string is taken to the height „b" and then released from rest in that position . Find the displacement of the string.

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{ --------------------- } (1)$$

The suitable solution of (1) is given by

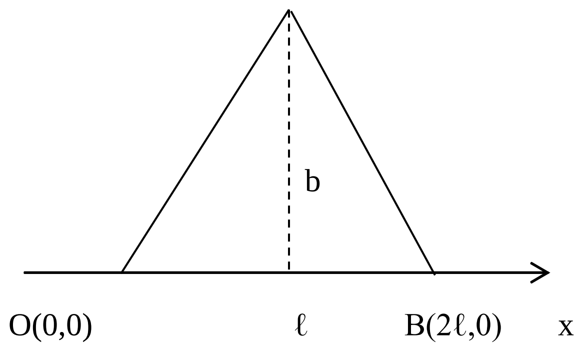
$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ----------- } (2)$$

The boundary conditions are

(i) $y(0,t) = 0, \forall\ t \geq 0.$

(ii) $y(\ell,t) = 0, \forall\ t \geq 0.$

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$, for $0 < x < 2\ell$.



$$\text{(iv) } y(x,0) = \begin{cases} (b/\ell)x, & 0<x<\ell \\ -(b/\ell)(x-2\ell), & \ell<x<2\ell \end{cases}$$

[Since, equation of OA is $y = (b/\ell)x$ and equation of AB is $(y-b)/(o-b) = (x-\ell)/(2\ell-\ell)$]

Using conditions (i) and (ii) in (2), we get

$$A = 0 \quad \& \quad \lambda = \frac{n\pi}{2\ell}$$

$$\therefore \quad y(x,t) = B \sin\frac{n\pi x}{2\ell}\left(C\cos\frac{n\pi at}{2\ell} + D\sin\frac{n\pi at}{2\ell}\right) \text{ -----------(3)}$$

$$\text{Now, } \frac{\partial y}{\partial t} = B \sin\frac{n\pi x}{2\ell}\left(-C\sin\frac{n\pi at}{2\ell} \cdot \frac{n\pi a}{2\ell} + D\cos\frac{n\pi at}{2\ell} \cdot \frac{n\pi a}{2\ell}\right)$$

Using (iii) in the above equation, we get

$$0 = B \sin\frac{n\pi x}{2\ell} \; D \; \frac{n\pi a}{2\ell}$$

Here B can not be zero, therefore D = 0.
Hence equation (3) becomes

$$y(x,t) = B \sin\frac{n\pi x}{2\ell} \cdot C\cos\frac{n\pi at}{2\ell}$$

nπx nπat

$$= B_1 \sin \frac{}{2\ell} . \quad \cos \frac{}{2\ell}, \text{ where } B_1 = BC$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2\ell} \cos\frac{n\pi a t}{2\ell} \text{ ------------------------ (4)}$$

Using (iv), We get

$$y(x,0) = \sum_{n=1}^{\infty} B_n . \sin\frac{n\pi x}{2\ell} \text{ -----------------------------(5)}$$

The RHS of equation (5) is the half range Fourier sine series of the LHS function .

$$\therefore B_n = \frac{2}{2\ell} \int_0^{2\ell} f(x) . \sin\frac{n\pi x}{2\ell} \, dx$$

$$= \frac{1}{\ell} \int_0^{\ell} f(x) . \sin\frac{n\pi x}{2\ell} \, dx + \int_{\ell}^{2\ell} f(x) . \sin\frac{n\pi x}{2\ell} \, dx$$

$$= \frac{1}{\ell} \int_0^{\ell} \frac{b}{\ell} x \sin\frac{n\pi x}{2\ell} \, dx + \int_{\ell}^{2\ell} \frac{-b}{\ell} (x-2\ell) \sin\frac{n\pi x}{2\ell} \, dx$$

$$= \frac{1}{\ell} \left\{ \frac{b}{\ell} \int_0^{\ell} x d\left(\frac{-\cos\frac{n\pi x}{2\ell}}{\frac{n\pi}{2\ell}} \right) - \frac{b}{\ell} \int_{\ell}^{2\ell} (x-2\ell) \, d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{2\ell}} \right) \right.$$

$$= \frac{1}{\ell} \left\{ \frac{b}{\ell} \left[(x) \left(\frac{-\cos\frac{n\pi x}{2\ell}}{\frac{n\pi}{2\ell}} \right) - (1) \left(\frac{-\sin\frac{n\pi x}{2\ell}}{\frac{n^2\pi^2}{4\ell^2}} \right) \right]_0^{\ell} - \right.$$

$$= \frac{b}{\ell^2}\left\{ \frac{-\ell\cos\frac{n\pi}{2}}{\frac{n\pi}{2\ell}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4\ell^2}} + \frac{\ell\cos\frac{n\pi}{2}}{\frac{n\pi}{2\ell}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4\ell^2}} \right\}$$

$$= \frac{8b\sin(n\pi/2)}{n^2\pi^2}$$

Therefore the solution is

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8b\sin(n\pi/2)}{n^2\pi^2}\cos\frac{n\pi at}{2\ell}\sin\frac{n\pi x}{2\ell}$$

Example 6

A tightly stretched string with fixed end points x = 0 & x = ℓ is initially in the position y(x,0) = f(x). It is set vibrating by giving to each of its points a velocity

$\frac{\partial y}{\partial t}$ = g(x) at t = 0 . Find the displacement y(x,t) in the form of Fourier series.

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{ -------------------- (1)}$$

The boundary conditions are

(i) $y(0,t) = 0, \forall\, t \geq 0.$

(ii) $y(\ell,t) = 0, \forall\, t \geq 0.$

(iii) $y(x,0) = f(x)$, for $0 \leq x \leq \ell$.

(iv) $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(x)$, for $0 \leq x \leq \ell$.

The solution of equation .(1) is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ------------(2)}$$

where A, B, C, D are constants.

Applying conditions (i) and (ii) in (2), we have

$$A = 0 \quad \text{and} \quad \lambda = \frac{n\pi}{\ell}.$$

Substituting in (2), we get

$$y(x,t) = B \sin \frac{n\pi x}{\ell} \left(C\cos \frac{n\pi at}{\ell} + D\sin \frac{n\pi at}{\ell}\right)$$

$$y(x,t) = \sin \frac{n\pi x}{\ell} \left(B_1\cos \frac{n\pi at}{\ell} + D_1 \sin \frac{n\pi at}{\ell}\right) \text{ where } B_1 = BC \text{ and } D_1 = BD.$$

The most general solution. is

$$y(x,t) = \sum_{n=1}^{\infty} \left[B_n \cos \frac{n\pi at}{\ell} + D_n .\sin \frac{n\pi at}{\ell}\right] .\sin \frac{n\pi x}{\ell} \text{ ------------------(3)}$$

Using (iii), we get

$$f(x) = \sum_{n=1}^{\infty} B_n . \sin \frac{n\pi x}{\ell} \quad \text{------------------------------} \; (4)$$

The RHS of equation (4) is the Fourier sine series of the LHS function.

$$\therefore\ B_n = \frac{2}{\ell} \int_0^{\ell} f(x) . \sin \frac{n\pi x}{\ell} \, dx$$

Differentiating (3) partially w.r.t „t", we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} -B_n \sin \frac{n\pi at}{\ell} \frac{n\pi a}{\ell} + D_n .\cos \frac{n\pi at}{\ell} \frac{n\pi a}{\ell} \ldots\ldots\ldots\ldots \sin \frac{n\pi x}{\ell}$$

Using condition (iv) , we get

$$g(x) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi a}{\ell} \right) . \sin \frac{n\pi x}{\ell} \quad \text{-----------------------} \; (5)$$

The RHS of equation (5) is the Fourier sine series of the LHS function.

$$\therefore\ D_n . \frac{n\pi a}{\ell} = \frac{2}{\ell} \int_0^{\ell} g(x) . \sin \frac{n\pi x}{\ell} \, dx$$

$$\Rightarrow D_n = \frac{2}{n\pi a} \int_0^{\ell} g(x) . \sin \frac{n\pi x}{\ell} \, dx$$

Substituting the values of B_n and D_n in (3), we get the required solution of the given equation.

Exercises

(1) Find the solution of the equation of a vibrating string of length „ℓ", satisfying the conditions

$y(0,t) = y(\ell,t) = 0$ and $y = f(x)$, $\partial y/\partial t = 0$ at $t = 0$.

(2) A taut string of length 20 cms. fastened at both ends is displaced from its position of equilibrium, by imparting to each of its points an initial velocity given by

$$v = x \quad \text{in } 0 \le x \le 10$$
$$= 20 - x \quad \text{in } 10 \le x \le 20,$$

„x" being the distance from one end. Determine the displacement at any subsequent time.

(3) Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

corresponding to the triangular initial deflection $f(x) = (2k/\ell)\, x$ when $0 < x < \ell/2$

$= (2k/\ell)(\ell - x)$ when $\ell/2 < x < \ell$,

and initial velocity zero.

(4) A tightly stretched string with fixed end points $x = 0$ and $x = \ell$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\partial y/\partial t = f(x)$ at $t = 0$. Find the displacement $y(x,t)$.

(5) Solve the following boundary value problem of vibration of string

i. $y(0,t) = 0$
ii. $y(\ell,t) = 0$
iii. $\frac{\partial y}{\partial t}(x,0) = x(x - \ell)$, $0 < x < \ell$.
iv. $y(x,0) = x$ in $0 < x < \ell/2$

$= \ell - x$ in $\ell/2 < x < \ell$.

(6) A tightly stretched string with fixed end points $x = 0$ and $x = \ell$ is initially in a position given by $y(x,0) = k(\sin(\pi x/\ell) - \sin(2\pi x/\ell))$. If it is released from rest, find the displacement of „y" at any distance „x" from one end at any time „t".

Solution of the heat equation

The heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{----------------(1).}$$

Let $u = X(x) \cdot T(t)$ be the solution of (1), where „X" is a function of „x" alone and „T" is a function of „t" alone.

Substituting these in (1), we get

$$X\,T' = \alpha^2 X''\,T.$$

i.e, $$\frac{X''}{X} = \frac{T'}{\alpha^2 T} \text{ ------------------------------ (2).}$$

Now the left side of (2) is a function of „x‟ alone and the right side is a function of „t‟ alone. Since „x‟ and „t‟ are independent variables, (2) can be true only if each side is equal to a constant.

Therefore, $$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = k \text{ (say).}$$

Hence, we get $X'' - kX = 0$ and $T' - \alpha^2 kT = 0$ --------------- (3).

Solving equations (3), we get

(i) when „k‟ is positive and $k = \lambda^2$, say

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

$$T = c_3 e^{\alpha^2 \lambda^2 t}$$

(ii) when „k‟ is negative and $k = -\lambda^2$, say

$$X = c_4 \cos\lambda x + c_5 \sin\lambda x$$

$$T = c_6 e^{-\alpha^2 \lambda^2 t}$$

(iii) when „k‟ is zero.

$$X = c_7 x + c_8$$
$$T = c_9$$

Thus the various possible solutions of the heat equation (1) are

$$u = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})\, c_3 e^{\alpha^2 \lambda^2 t} \text{ ---------------------- (4)}$$

$$u = (c_4 \cos\lambda x + c_5 \sin\lambda x)\, c_6 e^{-\alpha^2 \lambda^2 t} \text{ -------------- (5)}$$

$$u = (c_7 x + c_8)\, c_9 \text{ -- (6)}$$

Of these three solutions, we have to choose that solution which suits the physical nature of the problem and the given boundary conditions. As we are dealing with problems on heat flow, u(x,t) must be a transient solution such that „u" is to decrease with the increase of time „t".

Therefore, the solution given by (5),

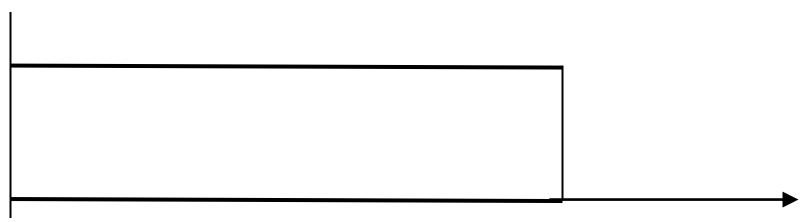
$$u = (c_4 \cos\lambda x + c_5 \sin \lambda x)\, c_6\, e^{-\alpha^2 \lambda^2 t}$$

is the only suitable solution of the heat equation.

Illustrative Examples

Example 7

A rod „ℓ" cm with insulated lateral surface is initially at temperature f(x) at an inner point of distance x cm from one end. If both the ends are kept at zero temperature, find the temperature at any point of the rod at any subsequent time.



Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{------------- (1)}$$

The boundary conditions are

(i) $u(0,t) = 0, \quad \forall\, t \geq 0$

(ii) $u(\ell,t) = 0, \quad \forall\, t \geq 0$

(iii) $u(x,0) = f(x), \ 0 < x < \ell$

The solution of equation (1) is given by

$$u(x,t) = (A \cos\lambda x + B \sin\lambda x)\, e^{-\alpha^2 \lambda^2 t} \qquad \text{------------------------------ (2)}$$

Applying condition (i) in (2), we have

$$0 = A.e^{-\alpha^2 \lambda^2 t} \quad \text{which gives } A = 0$$

$$\therefore u(x,t) = B \sin\lambda x \, e^{-\alpha^2\lambda^2 t} \quad \text{------------ (3)}$$

Applying condition (ii) in the above equation, we get $0 = B\sin\lambda\ell \, e^{-\alpha^2\lambda^2 t}$

$$\text{i.e,} \quad \lambda\ell = n\pi \quad \text{or} \quad \lambda = \frac{n\pi}{\ell} \quad \text{(n is an integer)}$$

$$\therefore u(x,t) = B \sin\frac{n\pi x}{\ell} \, e^{-\frac{n^2\pi^2\alpha^2}{\ell^2} t}$$

Thus the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \, e^{-\frac{n^2\pi^2\alpha^2}{\ell^2} t} \quad \text{------------ (4)}$$

By condition (iii),

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} = f(x).$$

The LHS series is the half range Fourier sine series of the RHS function.

$$\therefore B_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\frac{n\pi x}{\ell} \, dx$$

Substituting in (4), we get the temperature function

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{\ell}\int_0^{\ell} f(x) \sin\frac{n\pi x}{\ell} \, dx \right) \sin\frac{n\pi x}{\ell} \, e^{-\frac{n^2\pi^2\alpha^2}{\ell^2} t}$$

Example 8

The equation for the conduction of heat along a bar of length ℓ is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, neglecting radiation. Find an expression for u, if the ends of the bar are maintained at zero temperature and if, initially, the temperature is T at the centre of the bar and falls uniformly to

zero at its ends.

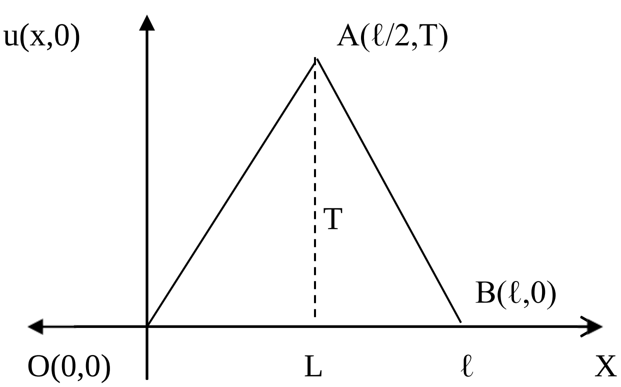
Let u be the temperature at P, at a distance x from the end A at time t.

The temperature function u (x,t) is given by the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, ---------- (1)

The boundary conditions are

(i) $u(0,t) = 0, \ \forall\, t \geq 0.$

(ii) $u(\ell,t) = 0, \ \forall\, t \geq 0.$



$$u(x,0) = \frac{2Tx}{\ell}, \text{ for } 0 \leq x \leq \frac{\ell}{2}$$

$$= \frac{2T}{\ell}(\ell - x), \text{ for } \frac{\ell}{2} \leq x \leq \ell$$

The solution of (1) is of the form

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \text{ ---------- (2)}$$

Applying conditions (i) and (ii) in (2), we get

$$A = 0 \ \& \ \lambda = \frac{n}{\ell}$$

$$\therefore u(x,t) = B \sin \frac{n\pi x}{\ell} \, e^{\frac{-n^2\pi^2\alpha^2}{\ell^2} t}$$

Thus the most general solution is

$$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\ell} \, e^{\frac{-n^2\pi^2\alpha^2}{\ell^2} t} \quad \text{------------------ (3)}$$

Using condition (iii) in (3), we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\ell} \quad \text{------------------ (4)}$$

We now expand u (x,0) given by (iii) in a half – range sine series in $(0,\ell)$

$$\text{Here } B_n = \frac{2}{\ell} \int_0^{\ell} u(x,0) \sin \frac{n\pi x}{\ell} \, dx$$

$$\text{ie, } B_n = \frac{2}{\ell} \left\{ \int_0^{\ell/2} \frac{2Tx}{\ell} \sin \frac{n\pi x}{\ell} \, dx + \int_{\ell/2}^{\ell} \frac{2T}{\ell} (\ell - x) \sin \frac{n\pi x}{\ell} \, dx \right\}$$

$$= \frac{4T}{\ell^2} \left\{ \int_0^{\ell/2} x \, d \left(\frac{-\cos \frac{n\pi x}{\ell}}{n\pi/\ell} \right) + \int_{\ell/2}^{\ell} (\ell - x) \, d \left(\frac{-\cos \frac{n\pi x}{\ell}}{n\pi/\ell} \right) \right\}$$

$$= \frac{4T}{\ell^2} \left[(x) \frac{-\cos \frac{n\pi x}{\ell}}{n\pi/\ell} - (1) \frac{-\sin \frac{n\pi x}{\ell}}{n^2\pi^2/\ell^2} \right]_0^{\ell/2} +$$

$$\left[(\ell - x) \; \frac{-\cos \frac{}{\ell}}{n\pi/\ell} \; - (-1) \; \frac{-\sin \frac{}{\ell}}{n^2\pi^2/\ell^2} \right]_{\ell/2}^{\ell}$$

$$= \frac{4T}{\ell^2} \left\{ \frac{-\ell^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{\ell^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{\ell^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{\ell^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right\}$$

$$= \frac{4T}{\ell^2} \; \frac{2\ell^2}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$\therefore B_n = \frac{8T}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8T}{n^2\pi^2} \sin \frac{n\pi}{2} \; \sin \frac{n\pi x}{\ell} \; e^{\frac{-n^2\pi^2\alpha^2}{\ell^2} t}$$

or

$$u(x,t) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8T}{n^2\pi^2} \sin \frac{n\pi}{2} \; \sin \frac{n\pi x}{\ell} \; e^{\frac{-n^2\pi^2\alpha^2}{\ell^2} t}$$

or

$$u(x,t) = \frac{8T}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{\ell} \; e^{\frac{-\alpha^2 (2n-1)^2\pi^2}{\ell^2} t}$$

Steady - state conditions and zero boundary conditions

Example 9

A rod of length „ℓ‟ has its ends A and B kept at 0°C and 100°C until steady state conditions prevails. If the temperature at B is reduced suddenly to 0°C and kept so while that of A is maintained, find the temperature u(x,t) at a distance x from A and at time „t‟.

The heat-equation is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{------------- (1)}$$

Prior to the temperature change at the end B, when t = 0, the heat flow was independent of time (steady state condition).

When the temperature u depends only on x, equation(1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0$$

Its general solution is $u = ax + b$ ------------------------- (2)

Since u = 0 for x = 0 & u = 100 for x = ℓ, therefore (2) gives b = 0 & $a = \frac{100}{\ell}$

$$\therefore u(x,0) = \frac{100}{\ell} x, \text{ for } 0 < x < \ell$$

Hence the boundary conditions are

(i) $u(0,t) = 0, \quad \forall\ t \geq 0$

(ii) $u(\ell,t) = 0, \quad \forall\ t \geq 0$

(iii) $u(x,0) = \frac{100x}{\ell}$, for $0 \leq x \leq \ell$

The solution of (1) is of the form

$$u(x,t) = (A \cos\lambda x + B \sin \lambda x)\ e^{-\alpha^2\lambda^2 t} \text{ ------------------------------------ (3)}$$

Using, conditions (i) and (ii) in (3), we get

$$A = 0 \ \&\ \lambda = \frac{n\pi}{\ell}$$

$$\therefore u(x,t) = B \sin \frac{n\pi x}{\ell}\ e^{\frac{-n^2\pi^2\alpha^2}{\ell^2} t}$$

Thus the most general solution is

$-n^2\pi^2\alpha^2$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}\ e^{\frac{-------t}{\ell^2}} \quad \text{-------------------------------}(4)$$

Applying (iii) in (4), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}$$

$$\text{ie, } \frac{100x}{\ell} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}$$

$$\Longrightarrow B_n = \frac{2}{\ell}\int_0^{\ell} \frac{100x}{\ell} \sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{200}{\ell^2}\int_0^{\ell} x \ d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right)$$

$$= \frac{200}{\ell^2}\left[(x)\ \frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}} - (1)\ \frac{-\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right]_0^{\ell}$$

$$= \frac{200}{\ell^2}\ \frac{-\ell^2}{n\pi}\cos n\pi$$

$$B_n = \frac{200\,(-1)^{n+1}}{n\pi}$$

Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{\ell} \, e^{\frac{-n^2\pi^2\alpha^2 t}{\ell^2}}$$

Example 10

A rod, 30 c.m long, has its ends A and B kept at 20°C and 80°C respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function u (x,t) taking x = 0 at A.

The one dimensional heat flow equation is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{------------ (1)}$$

In steady-state, $\frac{\partial u}{\partial t} = 0$.

Now, equation (1) reduces to $\frac{\partial^2 u}{\partial x^2} = 0$ -------------------(2)

Solving (2), we get $u = ax + b$ -----------------------------(3)

The initial conditions, in steady – state, are

u = 20, when x = 0
u = 80, when x = 30

Therefore, (3) gives b = 20, a = 2.

$\therefore u(x) = 2x + 20$------------------------------(4)

Hence the boundary conditions are

(i) $u(0,t) = 0, \quad \forall \ t \geq 0$
(ii) $u(30,t) = 0, \quad \forall \ t \geq 0$
(iii) $u(x,0) = 2x + 20$, for $0 < x < 30$

The solution of equation (1) is given by

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \quad \text{------------------ (5)}$$

Applying conditions (i) and (ii), we get

$$A = 0,\ \lambda = \frac{n\pi}{30}, \text{ where „n" is an integer}$$

$$\therefore\ u(x,t) = B \sin \frac{n\pi x}{30}\, e^{\frac{-\alpha^2 n^2 \pi^2}{900} t} \quad \text{---- (6)}$$

The most general solution is

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}\, e^{\frac{-\alpha^2 n^2 \pi^2}{900} t} \quad \text{------------- (7)}$$

Applying (iii) in (7), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} = 2x + 20,\ 0 < x < 30.$$

$$\therefore\ B_n = \frac{2}{30} \int_0^{30} (2x + 20) \sin \frac{n\pi x}{30}\, dx$$

$$= \frac{1}{15} \int_0^{30} (2x + 20)\, d \left(\frac{-\cos \frac{n\pi x}{30}}{\frac{n\pi}{30}} \right)$$

$$= \frac{1}{15} \left\{ (2x+20) \left(\frac{-\cos \frac{n\pi x}{30}}{\frac{n\pi}{30}} \right) - (2) \left(\frac{-\sin \frac{n\pi x}{30}}{\frac{n^2\pi^2}{900}} \right) \right\}_0^{30}$$

$$= \frac{1}{15}\left[\frac{-2400\cos n\pi \square 600}{n\pi} + \frac{}{n\pi}\right]$$

$$B_n = \frac{40}{n\pi}\{1 - 4(-1)^n\}$$

Hence, the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi \square 30}\{1 - 4(-1)^n\} \sin \frac{n\pi x}{} e^{\frac{-\alpha^2 n^2 \pi^2}{900} t}$$

Steady–state conditions and non–zero boundary conditions

Example 11

The ends A and B of a rod 30cm. long have their temperatures kept at 20°C and 80°C, until steady–state conditions prevail. The temperature of the end B is suddenly reduced to 60°C and kept so while the end A is raised to 40°C. Find the temperature distribution in the rod after time t.

Let the equation for the heat- flow be

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{-----------} (1)$$

In steady–state, equation (1) reduces to $\frac{\partial^2 u}{\partial x^2} = 0$.

Solving, we get $u = ax + b$ --------------------------------(2)

The initial conditions, in steady–state, are

u = 20, when x = 0
u = 80, when x = 30

From (2), b = 20 & a = 2.

Thus the temperature function in steady–state is

$$u(x) = 2x + 20 \text{ ---} (3)$$

Hence the boundary conditions in the transient–state are

(i) $u(0,t) = 40, \quad \forall\ t > 0$

(ii) $u(30,t) = 60, \quad \forall\ t > 0$

(iii) $u(x,0) = 2x + 20$, for $0 < x < 30$

we break up the required funciton u (x,t) into two parts and write

$$u(x,t) = u_s(x) + u_t(x,t) \text{ --} (4)$$

where $u_s(x)$ is a solution of (1), involving x only and satisfying the boundary condition (i) and (ii). $u_t(x,t)$ is then a function defined by (4) satisfying (1).

Thus $u_s(x)$ is a steady state solution of (1) and $u_t(x,t)$ may therefore be regarded as a transient solution which decreases with increase of t.

To find $u_s(x)$

we have to solve the equation $\dfrac{\partial^2 u}{\partial x^2} = 0$

Solving, we get $u_s(x) = ax + b$ -------------------------------(5)

Here $u_s(0) = 40$, $u_s(30) = 60$.

Using the above conditions, we get b = 40, a = 2/3.

$$\therefore u_s(x) = \frac{2}{3}x + 40 \text{ -----------------------------} (6)$$

To find $u_t(x,t)$

$$u_t(x,t) = u(x,t) - u_s(x)$$

Now putting x = 0 and x = 30 in (4), we have

$$u_t(0,t) = u(0,t) - u_s(0) = 40 - 40 = 0$$

and $$u_t(30,t) = u(30,t) - u_s(30) = 60 - 60 = 0$$

Also $u_t(x,0) = u(x,0) - u_s(x)$

$$= 2x + 20 - \frac{2}{3}\,40$$

$$= \frac{4}{3}x - 20$$

Hence the boundary conditions relative to the transient solution u_t (x,t) are

$$u_t(0,t) = 0 \quad \text{--------------------------- (iv)}$$

$$u_t(30,t) = 0 \quad \text{--------------------------- (v)}$$

and $$u_t(x,0) = (4/3)x - 20 \quad \text{-------------- (vi)}$$

We have

$$u_t(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \quad \text{-------------- (7)}$$

Using condition (iv) and (v) in (7), we get

$$A = 0 \ \& \ \lambda = \frac{n\pi}{30}$$

Hence equation (7) becomes

$$u_t(x,t) = B \sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2 \pi^2}{900} t}$$

The most general solution of (1) is

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2 \pi^2}{900} t} \quad \text{------------------- (8)}$$

Using condition (vi) ,

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30} = (4/3)x - 20, \ 0 < x < 30.$$

$$\therefore B_n = \frac{2}{30}\int_0^{30} \{(4/3)x - 20\} \sin\frac{n\pi x}{30}\, dx$$

$$= \frac{1}{15}\int_0^{30} \frac{4}{3}(x-20)\, d\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right)$$

$$= \frac{1}{15}\left\{\left(\frac{4}{3}(x-20)\right)\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right) - \frac{4}{3}\left(\frac{-\sin\frac{n\pi x}{30}}{\frac{n^2\pi^2}{900}}\right)\right\}$$

$$= \frac{1}{15}\left[\frac{-600\cos n\pi \square 600}{n\pi} - \frac{\quad}{n\pi}\right]$$

$$= \frac{-40}{n\pi}\{1 + \cos n\pi\}$$

$$B_n = \frac{-40\{1 + (-1)^n\}}{n\pi}$$

$$\text{or } B_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ \dfrac{-80}{n\pi}, & \text{when } n \text{ is even} \end{cases}$$

$$\therefore u_t(x,t) = \sum_{n=2,4,6,\ldots}^{\infty} \frac{-80}{n\pi} \sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2}{900}t}$$

$$\therefore u(x,t) = u_s(x) + u_t(x,t)$$

$$\text{ie, } u(x,t) = \frac{2}{3}x + 40 - \frac{80}{\pi}\sum_{n=2,4,6,..}^{\infty} \frac{1}{n}\sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2}{900}t}$$

Exercises

(1) Solve $\partial u/\partial t = \alpha^2 (\partial^2 u/\partial x^2)$ subject to the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$, $u(x,0) = x$, $0< x< l$.

(2) Find the solution to the equation $\partial u/\partial t = \alpha^2 (\partial^2 u/\partial x^2)$ that satisfies the conditions

i. $u(0,t) = 0$,
ii. $u(l,t) = 0$, $\forall\ t > 0$,
iii. $u(x,0) = x$ for $0< x< l/2$.
$= l - x$ for $l/2< x< l$.

(3) Solve the equation $\partial u/\partial t = \alpha^2 (\partial^2 u/\partial x^2)$ subject to the boundary conditions

i. $u(0,t) = 0$,
ii. $u(l,t) = 0$, $\forall\ t > 0$,
iii. $u(x,0) = kx(l - x)$, $k > 0$, $0 \le x \le l$.

(4) A rod of length „l" has its ends A and B kept at 0^o C and 120^o C respectively until steady state conditions prevail. If the temperature at Bis reduced to 0^o C and kept so while that of A is maintained, find the temperature distribution in the rod.

(5) A rod of length „l" has its ends A and B kept at 0^o C and 120^o C respectively until steady state conditions prevail. If the temperature at Bis reduced to 0^o C and kept so while 10^o C and at the same instant that at A is suddenly raised to 50^o C. Find the temperature distribution in the rod after time „t".

(6) A rod of length „l" has its ends A and B kept at 0^o C and 100^o C respectively until steady state conditions prevail. If the temperature of A is suddenly raised to 50^o C and that of B to
150^o C, find the temperature distribution at the point of the rod and at any time.

(7) A rod of length 10 cm. has the ends A and B kept at temperatures 30^o C and 100^o C, respectively until the steady state conditions prevail. After some time, the temperature at A is lowered to 20^o C and that of B to 40^o C, and then these temperatures are maintained. Find the subsequent temperature distribution.

(8) The two ends A and B of a rod of length 20 cm. have the temperature at 30^o C and 80^o C respectively until th steady state conditions prevail. Then the temperatures at the ends A and B are changed to 40^o C and 60^o C respectively. Find $u(x,t)$.

(9) A bar 100 cm. long, with insulated sides has its ends kept at 0^o C and 100^o C until steady state condition prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution

(10) Solve the equation $\partial u/\partial t = \alpha^2 (\partial^2 u / \partial x^2)$ subject to the conditions (i) „u‟ is not infinite
as $t \to \infty$ (ii) $u = 0$ for $x = 0$ and $x = \pi$, $\forall$ t (iii) $u = \pi x - x^2$ for $t = 0$ in $(0, \pi)$.

Solution of Laplace's equation(Two dimentional heat equation)

The Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u = X(x) . Y(y)$ be the solution of (1), where „X‟ is a function of „x‟ alone and „Y‟ is a function of „y‟ alone.

Then $\frac{\partial^2 u}{\partial x^2} = X'' Y$ and $\frac{\partial^2 u}{\partial y^2} = . X Y''$

Substituting in (1), we have

$$X'' Y + X Y'' = 0$$

$$\text{i.e,} \quad \frac{X''}{X} = -\frac{Y''}{Y} \text{ ----------(2).}$$

Now the left side of (2) is a function of „x‟ alone and the right side is a function of „t‟ alone. Since „x‟ and „t‟ are independent variables, (2) can be true only if each side is equal to a constant.

$$\text{Therefore,} \quad \frac{X''}{X} = -\frac{Y''}{Y} = k \text{ (say).}$$

Hence, we get $X'' - kX = 0$ and $Y'' + kY = 0$ ---------------(3).

Solving equations (3), we get

(i) when „k‟ is positive and $k = \lambda^2$, say

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$
$$Y = c_3 \cos\lambda y + c_4 \sin \lambda y$$

(ii) when „k‟ is negative and $k = -\lambda^2$, say

$$X = c_5 \cos\lambda x + c_6 \sin \lambda x$$
$$Y = c_7 e^{\lambda y} + c_8 e^{-\lambda y}$$

(iii) when „k" is zero.

$X = c_9 x + c_{10}$

$Y = c_{11} x + c_{12}$

Thus the various possible solutions of (1) are

$$u = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 \cos\lambda y + c_4 \sin \lambda y) \text{------------} (4)$$

$$u = (c_5 \cos\lambda x + c_6 \sin \lambda x)(c_7 e^{\lambda y} + c_8 e^{-\lambda y}) \text{-------------} (5)$$

$$u = (c_9 x + c_{10})(c_{11} x + c_{12}) \text{---------------------------------} (6)$$

Of these three solutions, we have to choose that solution which suits the physical nature of the problem and the given boundary conditions.

Example 12

An infinitely long uniform plate is bounded by two parallel edges x = 0 & x = ℓ and an end at right angles to them. The breadth of this edge y = 0 is ℓ and this edge is maintained at a temperature f (x). All the other 3 edges are at temperature zero. Find the steady state temperature at any interior point of the plate.

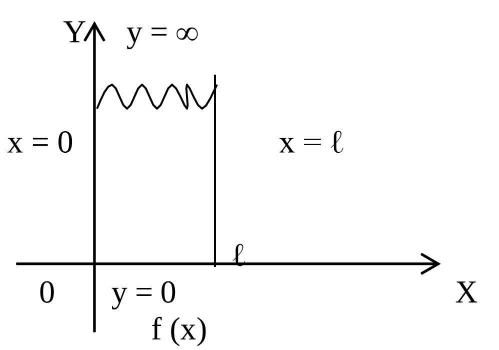
Solution

Let u (x,y) be the temperature at any point x,y of the plate.

Also u (x,y) satisfies the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ------------------------ (1)

Let the solution of equation (1) be

$$u(x,y) = (A \cos \lambda x + B \sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \text{-------------------------------------} (2)$$



$0 < x < \ell$

$0 < y < \infty$

The boundary conditions are

(i) $u(0, y) = 0$, for $0 < y < \infty$

(ii) $u(\ell, y) = 0$, for $0 < y < \infty$

(iii) $u(x, \infty) = 0$, for $0 < x < \ell$

(iv) $u(x, 0) = f(x)$, for $0 < x < \ell$

Using condition (i), we get

$$0 = A\,(Ce^{\lambda y} + De^{-\lambda y})$$

i.e, A = 0

∴ Equation (2) becomes,

$$u(x,y) = B \sin\lambda x\,(Ce^{\lambda y} + De^{-\lambda y}) \quad \text{-----} (3)$$

Using cndition (ii), we get

$$\lambda = \frac{n\pi}{\ell}$$

Therefore, $$u(x,y) = B \sin\frac{n\pi x}{\ell}\left\{ Ce^{(n\pi y/\ell)} + De^{(-n\pi y/\ell)} \right\} \quad \text{-----} (4)$$

Using condition (iii), we get C = 0.

$$\therefore u(x,y) = B \sin\frac{n\pi x}{\ell} De^{(-n\pi y/\ell)}$$

i.e, $u(x,y) = B_1 \sin\frac{n\pi x}{\ell}\, e^{(-n\pi y/\ell)}$, where $B_1 = BD$.

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}\, e^{(-n\pi y/\ell)} \quad \text{-----} (5)$$

Using condition (iv), we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \quad \text{-----} (6)$$

The RHS of equation (6) is a half – range Fourier sine series of the LHS function.

$$\therefore B_n = \frac{2}{\ell}\int_0^{\ell} f(x).\sin\frac{n\pi x}{\ell}\,dx \quad \text{-----} (7)$$

Using (7) in (5), we get the required solution.

Example 13

A rectangular plate with an insulated surface is 8 cm. wide and so long compared to its width that it may be considered as an infinite plate. If the temperature along short edge y = 0 is u(x,0) = 100 sin (πx/8), $0 \le x \le 8$, while two long edges x = 0 & x = 8 as well as the other short edges are kept at 0°C. Find the steady state temperature at any point of the plate.

Solution

The two dimensional heat equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{-------------- (1)}$$

The solution of equation (1) be

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \qquad \text{---------------------------------- (2)}$$

The boundary conditions are

(i) u (0, y) = 0, for $0 < y < \infty$
(ii) u (8, y) = 0, for $0 < y < \infty$
(iii) u (x, ∞) = 0, for $0 < x < 8$
(iv) u (x, 0) = 100 Sin (πx/8,) for $0 < x < 8$

Using conditions (i), & (ii), we get

$$A = 0,\ \lambda\ \frac{n\pi}{8}$$

$$\therefore u(x,y) = B\sin\frac{n\pi x}{8}\left(Ce^{(n\pi y/8)} + De^{(-n\pi y/8)}\right)$$

$$= \left(B_1e^{(n\pi y/8)} + D_1e^{(-n\pi y/8)}\right)\sin\frac{n\pi x}{8}, \text{ where } B_1 = BC,\ D_1 = BD$$

The most general soln is

$$u(x,y) = \sum_{n=1}^{\infty}\left(B_ne^{(n\pi y/8)} + D_ne^{(-n\pi y/8)}\right)\sin\frac{n\pi x}{8} \qquad \text{------------------------------(3)}$$

Using condition (iii), we get $B_n = 0$.

Hence, $u(x,y) = \sum_{n=1}^{\infty} D_n e^{(-n\pi y/8)} \sin\frac{n\pi x}{8}$ -- (4)

Using condition (iv), we get

$$100 \sin\frac{\pi x}{8} = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{8}$$

$$\text{i.e, } 100 \sin\frac{\pi x}{8} = D_1 \sin\frac{\pi x}{8} + D_2 \sin\frac{2\pi x}{8} + D_3 \sin\frac{3\pi x}{8} + \ldots$$

Comparing like coefficients on both sides, we get

$D_1 = 100$, $D_2 = D_3 = \ldots\ldots = 0$

Substituting in (4), we get

$$u(x,y) = 100\, e^{(-\pi y/8)} \sin(\pi x/8)$$

Example 14

A rectangular plate with an insulated surface 10 c.m wide & so long compared to its width that it may considered as an infinite plate. If the temperature at the short edge y = 0 is given by

$$u(x,0) = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$$

and all the other 3 edges are kept at temperature 0°C. Find the steady state temperature at any point of the plate.

Solution

The temperature function u (x,y) is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{--------------- (1)}$$

The solution is

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \qquad \text{---------------- (2)}$$

The boundary conditions are

(i) $u(0, y) = 0$, for $0 \leq y \leq \infty$

(ii) $u(10, y) = 0$, for $0 \leq y \leq \infty$

(iii) $u(x, \infty) = 0$, for $0 \leq x \leq 10$

(iv) $u(x, 0) = 20x$, if $0 \leq x \leq 5$

$20(10-x)$, if $5 \leq x \leq 10$

Using conditions (i), (ii), we get

$$A = 0 \ \& \ \lambda = \frac{n\pi}{10}$$

$\therefore$ Equation (2) becomes

$$u(x,y) = B \sin\frac{n\pi x}{10}\left(Ce^{(n\pi y/10)} + De^{(-n\pi y/10)}\right)$$

$$= \left(B_1 e^{(n\pi y/10)} + D_1 e^{(-n\pi y/10)}\right)\sin\frac{n\pi x}{10} \qquad \text{where } B_1 = BC,\ D_1 = BD$$

$\therefore$ The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty}\left(B_n e^{(n\pi y/10)} + D_n e^{(-n\pi y/10)}\right)\sin\frac{n\pi x}{10} \quad \text{------------} (3)$$

Using condition (iii), we get $B_n = 0$.

$\therefore$ Equation (3) becomes

$$u(x,y) = \sum_{n=1}^{\infty} D_n e^{(-n\pi y/10)}\sin\frac{n\pi x}{10} \quad \text{------------} (4)$$

Using condition (iv), we get

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{10} \qquad (5)$$

The RHS of equation (5) is a half range Fourier sine series of the LHS function

$$\therefore D_n = \frac{2}{10} \int_0^{10} f(x) \sin \frac{n\pi x}{10}\, dx$$

$$= \frac{2}{10} \left\{ \left[(20x) \frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} - (20) \frac{-\sin \frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}} \right]_0^5 + \left[20(10-x) \frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} - (-20) \frac{-\sin \frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}} \right]_5^{10} \right\}$$

i.e, $$D_n = \frac{800 \sin \frac{n\pi}{2}}{n^2\pi^2}$$

Substituting in (4) we get,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800 \sin \frac{n\pi}{2}}{n^2\pi^2} e^{(-n\pi y / 10)} \sin \frac{n\pi x}{10}$$

Example 15

A rectangular plate is bounded by the lines x = 0, x = a, y = 0 & y = b. The edge temperatures are u (0,y) = 0, u (x,b) = 0, u (a,y) = 0 &

$u(x,0) = 5 \sin(5\pi x / a) + 3 \sin(3\pi x / a)$. Find the steady state temperature distribution at any point of the plate.

The temperature function u (x,y) satisfies the equation

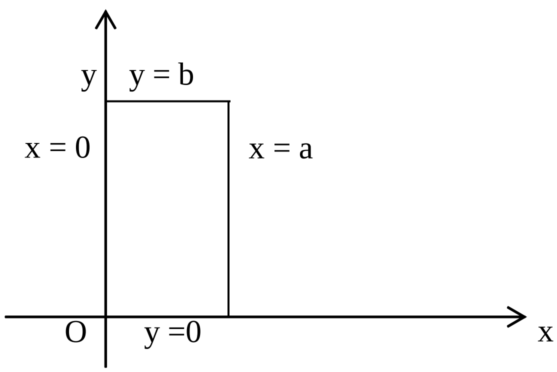
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{---------- (1)}$$

Let the solution of equation (1) be

$$u(x,y) = (A \cos\lambda x + B \sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \qquad \text{------------ (2)}$$

The boundary conditions are

(i) $u(0,y) = 0$, for $0 < y < b$
(ii) $u(a,y) = 0$, for $0 < y < b$
(iii) $u(x, b) = 0$, for $0 < x < a$
(iv) $u(x,0) = 5 \sin(5\pi x / a) + 3 \sin(3\pi x / a)$, for $0 < x < a$.



Using conditions (i), (ii), we get

$$A = 0, \ \lambda = \frac{n\pi}{a}$$

$$\therefore u(x,y) = B \sin \frac{n\pi x}{a} \left(Ce^{(n\pi y / a)} + De^{(-n\pi y / a)} \right)$$

$$= \sin \frac{n\pi x}{a} \left(B_1 e^{(n\pi y / a)} + D_1 e^{(-n\pi y / a)} \right)$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} \left(B_n e^{(n\pi y / a)} + D_n e^{(-n\pi y / a)} \right) \sin \frac{n\pi x}{a} \qquad \text{---(3)}$$

Using condition (iii) we get

$$0 = \sum_{n=1}^{\infty} \left(B_n e^{(n\pi b / a)} + D_n e^{(-n\pi b / a)} \right) \sin \frac{n\pi x}{a}$$

$$\Rightarrow B_n e^{(n\pi b / a)} + D_n e^{(-n\pi b / a)} = 0$$

$$\therefore \quad D_n = B_n \frac{e^{(n\pi b / a)}}{-e^{(-n\pi b / a)}} = - B_n e^{(2n\pi b / a)}$$

Substituting in (3), we get

$$u(x,y) = \sum_{n=1}^{\infty} \left(B_n e^{(n\pi y / a)} - B_n e^{(2n\pi b / a)} e^{(-n\pi y / a)} \right) \sin \frac{n\pi x}{a}$$

$$= \sum_{n=1}^{\infty} \frac{B_n}{e^{(-n\pi b)/a}} \left(e^{(n\pi y / a)} e^{(-n\pi b / a)} - e^{(2n\pi b / a)} e^{(-n\pi y / a)} e^{(-n\pi b / a)} \right) \sin \frac{n\pi x}{a}$$

$$= \sum \frac{2 B_n}{e^{(-n\pi b / a)}} \left(\frac{e^{(n\pi (y-b) / a)} - e^{(-n\pi (y-b) / a)}}{2} \right) \sin \frac{n\pi x}{a}$$

$$= \sum \frac{2B_n}{e^{(-n\pi b / a)}} \sinh \frac{n\pi (y-b)}{a} \sin \frac{n\pi x}{a}$$

$$\text{i.e,} \quad u(x,y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi (y-b)}{a} \sin \frac{n\pi x}{a} \quad \text{-----------(4)}$$

Using condition (iv), we get

$$5 \sin \frac{5\pi x}{a} + 3 \sin \frac{3\pi x}{a} = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi (-b)}{a} \sin \frac{n\pi x}{a}$$

$$\text{ie,} \quad 5 \sin \frac{5\pi x}{a} + 3 \sin \frac{3\pi x}{a} = \sum_{n=1}^{\infty} - C_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

ie, $5 \sin \frac{5\pi x}{a} + 3 \sin \frac{3\pi x}{a} = - C_1 \sinh \frac{\pi b}{a} \sin \frac{\pi x}{a} - C_2 \sin h \frac{2\pi b}{a} \sin \frac{2\pi x}{a} - \ldots$

Comparing the like coefficients on both sides, we get

$$- C_3 \sinh \frac{3\pi b}{a} = 3 \quad \&$$

$$- C_5 \sinh \frac{5\pi b}{a} = 5, \qquad C_1 = C_2 = C_4 = C_6 = . . . = 0$$

$$\Longrightarrow C_3 = \frac{-3}{\sinh (3\pi b /a)} \ \& \ C_5 = \frac{-5}{\sinh(5\pi b/ a)}$$

Substituting in (4), we get

$$u (x,y) = - \frac{3}{\sinh(3\pi b / a)} \sin h \frac{3\pi (y-b)}{a} \sin \frac{3\pi x}{a}$$

$$- \frac{5}{\sinh(5\pi b / a)} \sin h \frac{5\pi (y-b)}{a} \sin \frac{5\pi x}{a}$$

i.e, $$u (x,y) = \frac{3}{\sinh(3\pi b / a)} \sin h \frac{3\pi (b-y)}{a} \sin \frac{3\pi x}{a}$$

$$+ \frac{5}{\sinh(5\pi b / a)} \sin h \frac{5\pi (b-y)}{a} \sin \frac{5\pi x}{a}$$

Exercises

(1) Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions

i. $u(0,y) = 0$ for $0 < y < b$

ii. $u(a,y) = 0$ for $0 < y < b$
iii. $u(x,b) = 0$ for $0 < x < a$
iv. $u(x,0) = \sin^3(\pi x/a)$, $0 < x < a$.

(2) Find the steady temperature distribution at points in a rectangular plate with insulated faces and the edges of the plate being the lines x = 0, x = a, y = 0 and y = b. When three of the edges are kept at temperature zero and the fourth at a fixed temperature α^o C.

(3) Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, which satisfies the conditions $u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin(n\pi x/l)$.

(4) Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, which satisfies the conditions $u(0,y) = u(a,y) = u(x,b) = 0$ and $u(x,0) = x(a - x)$.

(5) Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions

i. $u(0,y) = 0$, $0 \le y \le l$ ii. $u(l,y) = 0$, $0 \le y \le l$
iii. $u(x,0) = 0$, $0 \le x \le l$ iv. $u(x,l) = f(x)$, $0 \le x \le l$

(6) A square plate is bounded by the lines x = 0, y = 0, x = 20 and y = 20. Its faces are insulated.
The temperature along the upper horizontal edge is given by $u(x,0) = x(20 - x)$, when $0 < x < 20$,
while other three edges are kept at 0^o C. Find the steady state temperature in the plate.

(7) An infinite long plate is bounded plate by two parallel edges and an end at right angles to them.The breadth is π. This end is maintained at a constant temperature „u_0" at all points and the other edges are at zero temperature. Find the steady state temperature at any point (x,y) of the plate.

(8) An infinitely long uniform plate is bounded by two parallel edges x = 0 and x = l, and an end at right angles to them. The breadth of this edge y = 0 is „l" and is maintained at a temperature f(x). All the other three edges are at temperature zero. Find the steady state temperature at any interior point of the plate.

(9) A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge y = 0 is given by $u(x,0) = 100 \sin(\pi x/8)$, $0 < x < 8$, while the two long edges x = 0 and x = 8 as well as the other short edge are kept at 0^o C, show that the steady state temperature at any point of the plane is given by $u(x,y) = 100\, e^{-\pi y/8} \sin \pi x/8$.

(10) A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite length. If the temperature along short edge $y = 0$ is given

$u(x,0) = 8 \sin(\pi x/10)$ when $0 < x < 10$, while the two long edges $x = 0$ and $x = 10$ as well as the other short edge are kept at 0^o C, find the steady state temperature distribution $u(x,y)$.

UNIT-IV

FOURIER TRANSFORMS

Introduction

This unit starts with integral transforms and presents three well-known integral transforms, namely, Complex Fourier transform, Fourier sine transform, Fourier cosine transform and their inverse transforms. The concept of Fourier transforms will be introduced after deriving the Fourier Integral Theorem. The various properties of these transforms and many solved examples are provided in this chapter. Moreover, the applications of Fourier Transforms in partial differential equations are many and are not included here because it is a wide area and beyond the scope of the book.

Integral Transforms

The **integral transform** $\tilde{f}(s)$ of a function f(x) is defined by

$$\tilde{f}(s) = \int_a^b f(x)\, K(s,x)\, dx,$$

if the integral exists and is denoted by I{f(x)}. Here, K(s,x) is called the **kernel** of the transform. The kernel is a known function of „s" and „x". The function f(x) is called the

inverse transform

of $\tilde{f}(s)$. By properly selecting the kernel in the definition of general integral transform, we get various integral transforms.

The following are some of the well-known transforms:

(i) Laplace Transform

$$L\{f(x)\} = \int_0^\infty f(x)\, e^{-sx}\, dx$$

(ii) Fourier Transform

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

(iii) Mellin Transform

$$M\{f(x)\} = \int_0^\infty f(x)\, x^{s-1}\, dx$$

(iv) Hankel Transform

$$H_n\{f(x)\} = \int_0^{\infty} f(x)\, x\, J_n(sx)\, dx,$$

where $J_n(sx)$ is the Bessel function of the first kind and order „n‟.

FOURIER INTEGRAL THEOREM

If f(x) is defined in the interval (-ℓ,ℓ), and the following conditions

(i) f(x) satisfies the Dirichlet‟s conditions in every interval (-ℓ,ℓ),

(ii) $\int_{-\infty}^{\infty} | f(x) |\, dx$ converges, i.e. f(x) is absolutely integrable in (-∞,∞)

are true, then $f(x) = (1/\pi) \int_0^{\infty}\int_{-\infty}^{\infty} f(t) \cos\lambda(t-x)\, dt\, d\lambda$.

Consider a function f(x) which satisfies the Dirichlet‟s conditions in every interval (-ℓ,ℓ) so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \quad (1)$$

where $a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\, dt$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t) \cos(n\pi t/\ell)\, dt$$

and $b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t) \sin(n\pi t/\ell)\, dt$

Substituting the values of a_0, a_n and b_n in (1), we get

$$f(x) = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(t)\, dt + \frac{1}{\ell}\sum_{n=1}^{\infty}\int_{-\ell}^{\ell} f(t) \cos\frac{n\pi(t-x)}{\ell}\, dt \quad \text{-------(2)}$$

Since, $\left| \dfrac{1}{2\ell} \int_{-\ell}^{\ell} f(t)\, dt \right| \leq \dfrac{1}{2\ell} \int_{-\ell}^{\ell} |f(t)|\, dt$,

then by assumption (ii), the first term on the right side of (2) approaches zero as $\ell \to \infty$. As $\ell \to \infty$, the second term on the right side of (2) becomes

$$\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi(t-x)}{\ell}\, dt$$

$$= \lim_{\Delta\lambda \to 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \Delta\lambda \int_{-\infty}^{\infty} f(t) \cos \{ n\, \Delta\lambda\, (t-x) \}\, dt$$,on taking $(\pi / \ell) = \Delta\lambda$.

By the definition of integral as the limit of sum and $(n\pi / \ell) = \lambda$ as $\ell \to \infty$, the second term of (2) takes the form

$$\frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda ,$$

Hence as $\ell \to \infty$, (2) becomes

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda \quad \text{---(3)}$$

which is known as the **Fourier integral** of f(x).

Note:

When f(x) satisfies the conditions stated above, equation (3) holds good at a point of continuity. But at a point of discontinuity, the value of the integral is (1/2) [f(x+0) + f(x-0)] as in the case of Fourier series.

Fourier sine and cosine Integrals

The Fourier integral of f(x) is given by

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda (t-x)\, dt\, d\lambda$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \{ \cos\lambda t \cdot \cos\lambda x + \sin\lambda t \cdot \sin\lambda x \}\, dt\, d\lambda$$

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$$= \frac{}{\pi} \int_{0} \cos\lambda x \int_{-\infty} f(t) \cos\lambda t \, dt \, d\lambda + \frac{}{\pi} \int_{0} \sin\lambda x \int_{-\infty} f(t) \sin\lambda t \, dt \, d\lambda \ldots (4)$$

When f(x) is an odd function, f(t) cosλt is odd while f(t) sinλt is even. Then the first integral of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \sin\lambda x \int_{-\infty}^{\infty} f(t) \sin\lambda t \, dt \, d\lambda \ldots (5)$$

which is known as the **Fourier sine integral.**

Similarly, when f(x) is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos\lambda x \int_{-\infty}^{\infty} f(t) \cos\lambda t \, dt \, d\lambda \ldots (6)$$

which is known as the **Fourier cosine integral.**

Complex form of Fourier Integrals

The Fourier integral of f(x) is given by

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t - x) \, dt \, d\lambda$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \int_{0}^{\infty} \cos \lambda(t - x) \, d\lambda \quad dt$$

Since cos λ(t – x) is an even function of λ, we have by the property of definite integrals

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \, (1/2) \int_{-\infty}^{\infty} \cos \lambda(t - x) \, d\lambda \, dt$$

$$\text{i.e.,} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t - x) \, dt \, d\lambda \ldots (7)$$

Similarly, since sin λ(t – x) is an odd function of λ, we have

$$0 = \frac{}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \lambda(t - x)\, dt\, d\lambda \quad (8)$$

Multiplying (8) by „i " and adding to (7), we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t - x)}\, dt\, d\lambda \quad (9)$$

which is the **complex form of the Fourier integral.**

Fourier Transforms and its properties

Fourier Transform

We know that the complex form of Fourier integral is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}\, dt\, d\lambda.$$

Replacing λ by s, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\, .$$

It follows that if

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \text{-----------------} (1)$$

Then, $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \text{ ----------------- } (2)$$

The function F(s), defined by (1), is called the **Fourier Transform** of f(x). The function f(x), as given by (2), is called the **inverse Fourier Transform** of F(s). The equation (2) is also referred to as the **inversion formula**.

Properties of Fourier Transforms

(1) Linearity Property

If F(s) and G(s) are Fourier Transforms of f(x) and g(x) respectively, then

$$F\{a f(x) + bg(x)\} = a F(s) + bG(s),$$

where a and b are constants.

We have $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$G(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

Therefore,

$$F\{a f(x) + b g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \{a f(x) + bg(x)\}dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

$$= a F(s) + bG(s)$$

i.e, $F\{a f(x) + bg(x)\} = a F(s) + bG(s)$

(2) Shifting Property

(i) If F(s) is the complex Fourier Transform of f(x), then

$$F\{f(x-a)\} = e^{isa} F(s).$$

We have $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{------------------(i)}$$

Now, $$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx$$

Putting x-a = t, we have

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(t+a)} f(t)\, dt.$$

$$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\, dt.$$

$$= e^{ias} . F(s). \qquad \text{(by (i)).}$$

(ii) If F(s) is the complex Fourier Transform of f(x), then

$$F\{e^{iax} f(x)\} = F(s+a).$$

We have
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{------------------ (i)}$$

Now,
$$F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} . e^{iax} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x.} f(x)\, dx .$$

$$= F(s+a) \qquad \text{by (i) .}$$

(3) Change of scale property

If F(s) is the complex Fourier transform of f(x), then

$$F\{f(ax)\} = 1/a\ F(s/a),\ a \neq 0.$$

We have
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{ ----------------- (i)}$$

Now,
$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(ax)\, dx.$$

Put ax = t, so that dx = dt/a.

$$\therefore F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\ ist/a} .f(t)\, dt/a .$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} . \int_{-\infty}^{\infty} e^{i(s/a)t} f(t)\, dt .$$

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$$= \frac{\;}{a} \,.\, F(s/a). \qquad \text{(by (i)).}$$

(4) Modulation theorem.

If F(s) is the complex Fourier transform of f(x),

Then $\qquad F\{f(x) \cos ax\} = ½\{F(s+a) + F(s-a)\}$.

We have $\qquad F(s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

Now, $\qquad F\{f(x) \cos ax\} = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{isx} .f(x) \cos ax.\, dx.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx.}\, f(x) \frac{e^{iax} + e^{-iax}}{2}\, dx\ .$$

$$= \frac{1}{2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} .f(x)\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\, dx \right.$$

$$= \frac{1}{2} \{ F(s+a) + F(s-a)\}$$

(5) n^{th} derivative of the Fourier Transform

If F(s) is the complex Fourier Transform of f(x),

Then $\quad F\{x^n f(x)\} = (-i)^n\, d^n/ds^n .F(s).$

We have $\qquad F(s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$---------------------(i)

Differentiating (i) „n" times w.r.t „s", we get

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n . e^{isx} f(x)\, dx$$

$$= \frac{(i)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \{x^n f(x)\}\, dx$$

$$= (i)^n F\{x^n f(x)\}.$$

$$\Rightarrow F\{x^n f(x)\} = \frac{1}{(i)^n} . \frac{d^n F(s)}{ds^n}$$

$$\text{i.e, } F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s).$$

(6) Fourier Transform of the derivatives of a function.

If F(s) is the complex Fourier Transform of f(x),

Then, $F\{f''(x)\} = -is F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$.

We have $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx .$$

Now, $$F\{f''(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f''(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} d\{f(x)\}.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx}.f(x) \right]_{-\infty}^{\infty} - is \int_{-\infty}^{\infty} f(x). e^{isx}\, dx. \right.$$

$$= -is \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{ , provided } f(x) = 0 \text{ as } x \to \pm\infty .$$

$$= - is F(s).$$

i.e, $F\{f'(x)\} = - is F(s)$--------------------- (i)

Then the Fourier Transform of $f''(x)$,

$$\text{i.e, } F\{f''(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f''(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, d\{f'(x)\}.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx}.f'(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x).\, e^{isx}.(is)dx. \right.$$

$$= -is \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx \text{ , provided } f'(x) = 0 \text{ as } x \to \pm\infty .$$

$$= - is F\{f'(x).\}$$

$$= (-is).(-is)F(s). \quad \text{by(i).}$$

$$= (-is)^2 . F(s).$$

i.e, $F\{f''(x)\} = (-is)^2 .F(s)$, Provided $f , f' \to 0$ as $x \to \pm\infty$.

In general, the Fourier transform of the n^{th} derivative of f(x) is given by

$$F\{f^n(x)\} = (-is)^n F(s),$$

provided the first 'n-1' derivatives vanish as $x \to \pm\infty$.

Property (7)

If F(s) is the complex Fourier Transform of f(x), then $F \int_a^x f(x)dx = \frac{F(s)}{(-is)}$

Let $g(x) = \int_a^x f(x)\, dx$.

Then, $g''(x) = f(x)$.-------------------------------------- (i)

Now $f[g_{,,}(x)] = (-is)\, G(s)$, by property (6).

$= (-is).\ F\{g(x)\}$

$= (-is).\ F \int_a^x f(x)\, dx$.

i.e, $F\{g''(x)\} = (-is).\ F \int_a^x f(x)\, dx$.

i.e, $F \int_a^x f(x)\, dx = \dfrac{1}{(-is)} .\ F\{g''(x)\}$.

$= \dfrac{1}{(-is)} F\{f(x)\}$.   [by (i)]

Thus, $F \int_a^x f(x)\, dx = \dfrac{F(s)}{(-is)}$.

Property (8)

If F(s) is the complex Fourier transform of f(x),

Then, $F\{\overline{f(-x)}\} = \overline{F}(s)$, where bar denotes complex conjugate.

Proof

$$\overline{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(x)\, e^{-isx}\, dx .$$

Putting x = -t, we get

$$\overline{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(-t)\, e^{isx}\, dt .$$

$$= F\{\overline{f(-x)}\} .$$

Note: If F{f(x)} = F(s), then

(i) $F\{f(-x)\} = F(-s)$.

(ii) $F\{\overline{f(x)}\} = \overline{F(-s)}$.

Example 1

Find the F.T of f(x) defined by

$$f(x) = \begin{cases} 0 & x<a \\ 1 & a<x<b \\ 0 & x>b. \end{cases}$$

The F.T of f(x) is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{isx} .dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{isx}}{is} \right)_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{ibs} - e^{ias}}{is}$$

Example 2

Find the F.T of $f(x) = x$ for $| x | \leq a$
$= 0$ for $| x | > a$.

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} .x.dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x \, d\left(\frac{e^{isx}}{is}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{xe^{isx}}{is} - \frac{e^{isx}}{(is)^2} \right\}_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{ae^{isa}}{is} - \frac{e^{isa}}{(is)^2} + \frac{ae^{-isa}}{is} + \frac{e^{-isa}}{(is)^2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{a}{is} (e^{isa} + e^{-isa}) + \frac{1}{s^2} (e^{isa} - e^{-isa}) \right.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{-2ai}{s} \text{cossa} + \frac{2i}{s^2} \text{sinsa} \right.$$

$$= \frac{2i}{s^2} \cdot \frac{1}{\sqrt{2\pi}} [\text{sinsa - as cossa}].$$

$$= \sqrt{(2/\pi)} \frac{i\,[\text{sinsa - as cossa}]}{s^2}$$

Example 3

Find the F.T of $f(x) = e^{iax}, 0 < x < 1$

$= 0$ otherwise

The F.T of f(x) is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$= \frac{1}{} \int^{1} e^{isx} . e^{iax} dx.$$

$\sqrt{2\pi} \quad 0$

$$= \frac{1}{\sqrt{2\pi}} \int_0^1 e^{i(s+a)x} .dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i(s+a)x}}{i(s+a)} \right)_0^1$$

$$= \frac{1}{i\sqrt{2\pi}.(s+a)} \{e^{i(s+a)x} - 1\}$$

$$= \frac{i}{\sqrt{2\pi}.(s+a)} \{1 - e^{i(s+a)}\}$$

Example 4

Find the F.T of $e^{-a^2 x^2}$, a>0 and hence deduce that the F.T of $e^{-x^2/2}$ is $e^{-s^2/2}$.

The F.T of f(x) is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$F\, e^{-a^2x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} . e^{isx} .dx.$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[ax - (is/2a)]^2} dx .$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt, \text{ by putting } ax - (is/2a) = t$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} . \sqrt{\pi} , \text{ since } \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \text{ (using Gamma functions).}$$

$$= \frac{1}{\sqrt{2}.a} e^{-s^2/4a^2} \text{ . ------------------- (i)}$$

To find $F\{e^{-x^2/2}\}$

Putting $a = 1/\sqrt{2}$ in (1), we get

$$F\{e^{-x^2/2}\} = e^{-s^2/2} .$$

Note:

If the F.T of f(x) is f(s), the function f(x) is called self-reciprocal. In the above example $e^{-x^2/2}$ is self-reciprocal under F.T.

Example 5

Find the F.T of

f(x) = 1 for |x|<1.

= 0 for |x|>1.

Hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$.

The F.T of f(x),

$$\text{i.e.,} \quad F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx} .(1).dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{isx}}{is} \right)_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} . \frac{e^{is} - e^{-is}}{is}$$

$$= \sqrt{(2/\pi)} \frac{\sin s}{s} \quad , s \neq 0$$

Thus, $F\{f(x)\} = F(s) = \sqrt{(2/\pi)}.\frac{\sin s}{s}$, $s \neq 0$

s

Now by the inversion formula , we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s).\ e^{-isx}\ .ds.$$

or

$$= \int_{-\infty}^{\infty} \sqrt{(2/\pi)} \frac{\sin s}{s}.\ e^{-isx}\ .ds. = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1. \end{cases}$$

$$\text{i.e, } \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} e^{-isx}\ .\ ds. = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1. \end{cases}$$

Putting x = 0, we get

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = 1$$

$$\text{i.e, } \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin s}{s} ds = 1, \text{ since the integrand is even.}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

$$\text{Hence, } \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Exercises

(1) Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x|<a \\ 0 & \text{for } |x|>a. \end{cases}$$

(2) Find the Fourier transform of

$$f(x) = \begin{cases} x^2 & \text{for } |x|\le a \\ 0 & \text{for } |x|>a. \end{cases}$$

(3) Find the Fourier transform of

$$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0. \end{cases}$$

Hence deduce that $$\int_{-\infty}^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$$

(4) Find the Fourier transform of $e^{-a|x|}$ and $x\, e^{-a|x|}$. Also deduce that

$$\int_{-\infty}^{\infty} \frac{\cos xt}{a^2 + t^2}\, dt = \frac{\pi}{2a} e^{-a|x|}$$

$$\{\text{Hint}: F\{x.\, e^{-a|x|}\} = -i \frac{d}{ds} F\{ e^{-a|x|}\}\}$$

Convolution Theorem and Parseval's identity.

The convolution of two functions f(x) and g(x) is defined as

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, g(x-t).\, dt.$$

Convolution Theorem for Fourier Transforms.

The Fourier Transform of the convolution of f(x) and g(x) is the product of their Fourier Transforms,

i.e, $F\{f(x) * g(x)\} = F\{f(x).F\{g(x)\}$.

Proof:

$F\{f(x) * g(x)\} = F\{(f*g)x)\}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x).\, e^{isx}\, .\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, g(x-t).\, dt\, e^{isx}\, dx \right. .$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t).\, e^{isx}\, dx \,.\; dt. \right.$$

(by changing the order of integration).

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\,(t).F\{g(x-t)\}.\, dt.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{its}\, .G(s).\, dt. \text{ (by shifting property)}$$

$$= G(s). \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{ist}\, dt.$$

$$= F(s).G(s).$$

Hence, $F\{f(x) * g(x)\} = F\{f(x).F\{g(x)\}.$

Parseval's identity for Fourier Transforms

If F(s) is the F.T of f(x), then

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds.$$

Proof:

By convolution theorem, we have

$$F\{f(x) * g(x)\} = F(s).G(s).$$

Therefore, $(f*g)\,(x) = F^{-1}\{F(s).G(s)\}.$

$$\text{i.e, } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, g(x-t).\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s).G(s).e^{-isx}\, ds \text{ ---------- (1)}$$

(by using the inversion formula)

Putting x = 0 in (1) , we get

$$\int_{}^{\infty} f(t).\, g(-t).\, dt = \int_{}^{\infty} F(s).G(s).ds \text{ ----------- (2)}$$

$-\infty \qquad\qquad -\infty$

Since (2) is true for all g(t), take $g(t) = \overline{f(-t)}$ and hence $g(-t) = \overline{f(t)}$ ---------(3)

Also, G(s) = F{g(t)}

$= F\{\overline{f(-t)}\}$

$= \overline{F(s)}$ ---------------- (4) (by the property of F.T).

Using (3) & (4) in (2), we have

$$\int_{-\infty}^{\infty} f(t).\overline{f(t)}.\,dt = \int_{-\infty}^{\infty} F(s).\overline{F(s)}.ds.$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$$

$$\text{i.e, } \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$$

Example 6

Find the F.T of f (x) = 1-|x| for |x | < 1.

= 0 for |x| > 1

and hence find the value $\int_0^{\infty} \frac{\sin^4 t}{t^4}\,dt.$

$$\text{Here, } F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)e^{isx}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\cos sx\,dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\sin sx\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_0^1 (1-x)\cos sx\, dx. \text{ by the property of definite integral.}$$

$$= \sqrt{(2/\pi)} \int_0^1 (1-x)\, d\left(\frac{\sin sx}{s}\right)$$

$$= \sqrt{(2/\pi)}\ \left\{ (1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(- \frac{\cos sx}{s^2}\right) \right\}_0^1$$

$$= \sqrt{(2/\pi)} \left(\frac{1-\cos s}{s^2}\right)$$

Using Parseval"s identity, we get

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds. = \int_{-1}^{1}(1-|x|)^2\, dx.$$

$$\Rightarrow \frac{4}{\pi}\int_0^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds. = 2\int_0^1 (1-x)^2\, dx = 2/3.$$

$$\text{i.e, } \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4(s/2)}{s^4}\, ds. = 2/3.$$

Setting s/2 = x , we get

$$\frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 x}{16x^4}\, 2.dx. = 2/3.$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^4 x}{x^4}\, dx. = \pi/3.$$

Example 7

Find the F.T of f(x) if

$$f(x) = \begin{cases} 1 & \text{for } |x|<a \\ 0 & \text{for } |x|>a>0. \end{cases}$$

Using Parseval‟s identity, prove $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt. = \pi/2.$

Here,

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} .(1) .dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{isx}}{is} \right)_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \quad \frac{e^{isa} - e^{isa}}{is}$$

$$= (\sqrt{2/\pi}) \frac{\sin as}{s}$$

i.e., $F(s) = (\sqrt{2/\pi}) \dfrac{\sin as}{s}$.

Using Parseval‟s identity

$$\int_{-\infty}^{\infty} | f (x) |^2 dx = \int_{-\infty}^{\infty} | F(s) |^2 ds,$$

we have

$$\int_{-a}^{a} 1 . dx = \int_{-\infty}^{\infty} (2/\pi) \left(\frac{\sin as}{s} \right)^2 ds.$$

$$2a = (2/\pi) \int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right)^2 ds.$$

Setting as = t, we get

$$(2/\pi)\int_{-\infty}^{\infty}\left(\frac{\sin t}{(t/a)}\right)^2 dt/a = 2a$$

i.e., $$\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \pi$$

$$\Rightarrow \quad 2\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \pi$$

Hence, $$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \pi/2.$$

Fourier sine and cosine transforms:

Fourier sine Transform

We know that the Fourier sine integral is

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\lambda x\,.\int_0^{\infty} f(t)\sin\lambda t\ dt.d\lambda.$$

Replacing λ by s, we get

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin sx\,.\int_0^{\infty} f(t)\sin st\ dt.\ ds.$$

It follows that if

$$F_s(s) = \sqrt{(2/\pi)}\int_0^{\infty} f(t)\sin st\ dt..$$

i.e., $$F_s(s) = \sqrt{(2/\pi)}\int_0^{\infty} f(x)\sin sx\ dx. \text{----------------------}(1)$$

then $$f(x) = \sqrt{(2/\pi)}\int_0^{\infty} F_s(s)\sin sx\ ds. \text{---------------------}(2)$$

The function $F_s(s)$, as defined by (1), is known as the **Fourier sine transform** of f(x). Also the function f(x), as given by (2),is called the **Inverse Fourier sine transform** of $F_s(s)$.

Fourier cosine transform

Similarly, it follows from the Fourier cosine integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \,.\, \int_0^\infty f(t) \cos\lambda t \; dt.d\lambda.$$

that if $F_c(s) = \sqrt{(2/\pi)} \int_0^\infty f(x) \cos sx \; dx.$ ---------------------- (3)

then $f(x) = \sqrt{(2/\pi)} \int_0^\infty F_c(s) \cos sx \; ds.$ ---------------------------- (4)

The function $F_c(s)$, as defined by (3), is known as the **Fourier cosine transform** of f(x). Also the function f(x), as given by (4),is called the **Inverse Fourier cosine transform** of $F_c(s)$.

Properties of Fourier sine and cosine Transforms

If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of f(x) respectively, the following properties and identities are true.

(1) Linearity property

$$F_s [a f(x) + b g(x)] = a F_s \{ f(x) \} + b F_s \{ g(x) \}.$$

and $$F_c [a f(x) + b g(x)] = a F_c \{ f(x) \} + b F_c \{ g(x) \}.$$

(2) Change of scale property

$$F_s [f(ax)] = (1/a) F_s[s/a].$$

and $$F_c [f(ax)] = (1/a) F_c[s/a].$$

(3) Modulation Theorem

i. $F_s [f(x) \sin ax] = (1/2) [F_c (s-a) - F_c (s+a)].$

ii. $F_s [f(x) \cos ax] = (1/2) [F_s (s+a) + F_s (s-a)].$

iii. $F_c[\, f(x) \cos ax \,] = (1/2)\,[\, F_c\,(s+a) + F_c\,(s-a)\,]$.

iv. $F_c[\, f(x) \sin ax \,] = (1/2)\,[\, F_s\,(s+a) - F_s\,(s-a)\,]$.

Proof

The Fourier sine transform of f(x)sinax is given by

$$F_s\,[\, f(x) \sin ax \,] = \sqrt{(2/\pi)} \int_0^\infty (f(x) \sin ax) \sin sx \, dx.$$

$$= (1/2)\, \sqrt{(2/\pi)} \int_0^\infty f(x)\, [\cos(s-a)x - \cos(s+a)x]\, dx.$$

$$= (1/2)\, [\, F_c\,(s-a) - F_c\,(s+a)\,].$$

Similarly, we can prove the results (ii), (iii) & (iv).

(4) Parseval's identity

$$\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx .$$

$$\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx .$$

$$\int_0^\infty \left| F_c(s) \right|^2 ds = \int_0^\infty \left| f(x) \right|^2 dx .$$

$$\int_0^\infty \left| F_s(s) \right|^2 ds = \int_0^\infty \left| f(x) \right|^2 dx .$$

Proof

$$\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty F_c(s)\, [\sqrt{(2/\pi)} \int_0^\infty g(t) \cos st\, dt]\, ds$$

$$= \int_0^\infty g(t)\, [\sqrt{(2/\pi)} \int_0^\infty F_c(s) \cos st\, ds]\, dt$$

$$= \int^\infty g(t)\, f(t)\, dt$$

i.e., $\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$.

Similarly, we can prove the second identity and the other identities follow by setting g(x) = f(x) in the first identity.

Property (5)

If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of f(x) respectively, then

(i) $F_s\{ x f(x) \} = -\dfrac{d}{ds} F_c(s)$.

(ii) $F_c\{ x f(x) \} = -\dfrac{d}{ds} F_s(s)$.

Proof

The Fourier cosine transform of f(x),

i.e., $F_c(s) = \sqrt{(2/\pi)} \int_0^\infty f(x) \cos sx\, dx.$

Differentiating w.r.t s, we get

$$\frac{d}{ds} [F_c(s)] = \sqrt{(2/\pi)} \int_0^\infty f(x) \{- x \sin sx \}\, dx.$$

$$= - \sqrt{(2/\pi)} \int_0^\infty (x f(x)) \sin sx\, dx.$$

$$= - F_s\{x f(x)\}$$

i.e., $F_s\{x f(x)\} = -\dfrac{d}{ds} \{ F_c(s) \}$

Similarly, we can prove

$$F_c\{x f(x)\} = -\frac{d}{ds} \{ F_s(s) \}$$

Example 8

Find the Fourier sine and cosine transforms of e^{-ax} and hence deduce the inversion formula.

The Fourier sine transform of f(x) is given by

$$F_s \{ f(x) \} = \sqrt{(2/\pi)} \int_0^\infty f(x) \sin sx\, dx.$$

Now , $F_s\{e^{-ax}\} = \sqrt{(2/\pi)} \int_0^\infty e^{-ax} \sin sx \, dx.$

$$= \sqrt{(2/\pi)} \left\{ \frac{e^{-ax}(-a \sin sx - s \cos sx)}{a^2 + s^2} \right\}_0^\infty$$

$$= \sqrt{(2/\pi)} \frac{s}{a^2 + s^2} \quad , \text{if } a>0$$

The Fourier cosine transform of f(x) is given by

$$F_c\{f(x)\} = \sqrt{(2/\pi)} \int_0^\infty f(x) \cos sx \, dx.$$

Now , $F_c\{e^{-ax}\} = \sqrt{(2/\pi)} \int_0^\infty e^{-ax} \cos sx \, dx.$

$$= \sqrt{(2/\pi)} \left\{ \frac{e^{-ax}(-a \cos sx + s \sin sx)}{a^2 + s^2} \right\}_0^\infty$$

$$= \sqrt{(2/\pi)} \frac{a}{a^2 + s^2} \quad , \text{if } a>0$$

Example 9

Find the Fourier cosine transform of $f(x) = \begin{cases} x, & \text{for } 0<x<1 \\ 2-x, & \text{for } 1<x<2 \\ 0, & \text{for } x>2 \end{cases}$

The Fourier cosine transform of f(x),

$$\text{i.e., } F_c\{f(x)\} = \sqrt{(2/\pi)} \int_0^1 x \cos sx \, dx. + \sqrt{(2/\pi)} \int_1^2 (2-x) \cos sx \, dx.$$

$$= \sqrt{(2/\pi)} \int_0^1 x \, d\left(\frac{\sin sx}{s}\right) + \sqrt{(2/\pi)} \int_1^2 (2-x) \, d\left(\frac{\sin sx}{s}\right)_1$$

$$= \sqrt{(2/\pi)} \; x\left(\frac{\sin sx}{s}\right) - (1) \; - \frac{\cos sx}{s^2} \quad {}^{0} \quad {}^{2}$$

$$+ \sqrt{(2/\pi)} \; (2-x)\left(\frac{\sin sx}{s}\right) - (-1) + - \frac{\cos sx}{s^2}$$

$$= \sqrt{(2/\pi)}\left\{\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right)\right\}$$

$$= \sqrt{(2/\pi)}\left(\frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}\right)$$

Example 10

Find the Fourier sine transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi e^{-m}}{2}$, m>0.

The Fourier sine transform of f(x) is given by

$$F_s\{ f(x) \} = \sqrt{(2/\pi)} \int_0^\infty f(x) \sin sx \; dx.$$

$$= \sqrt{(2/\pi)} \int_0^\infty e^{-x} \sin sx \; dx.$$

$$= \sqrt{(2/\pi)} \left\{ \frac{e^{-x} (- \sin sx - s \cos sx)}{1 + s^2} \right\}_0^\infty$$

$$= \sqrt{(2/\pi)} \frac{s}{1 + s^2}.$$

Using inversion formula for Fourier sine transforms, we get

$$\sqrt{(2/\pi)} \int_0^\infty \left\{ \sqrt{(2/\pi)} \frac{s}{1 + s^2} \right\} \sin sx \; ds. = e^{-x}$$

Replacing x by m,

$$e^{-m} = (2/\pi) \int_0^\infty \frac{s \sin ms}{1 + s^2} \; ds$$

∞ x sinmx

$$= (2/\pi) \int_0 \frac{}{1 + x^2} dx$$

Hence, $$\int_0^\infty \frac{x \sin mx}{1 + x^2} dx = \frac{\pi e^{-m}}{2}$$

Example 11

Find the Fourier sine transform of $\frac{x}{a^2+x^2}$ and the Fourier cosine transform of $\frac{1}{a^2+x^2}$.

To find the Fourier sine transform of $\frac{x}{a^2+x^2}$,

We have to find $F_s \{ e^{-ax} \}$.

Consider, $$F_s \{ e^{-ax} \} = \sqrt{(2/\pi)} \int_0^\infty e^{-ax} \sin sx \, dx.$$

$$= \sqrt{(2/\pi)} \frac{s}{a^2 + s^2} .$$

Using inversion formula for Fourier sine transforms, we get

$$e^{-ax} = \sqrt{(2/\pi)} \int_0^\infty \sqrt{(2/\pi)} \left. \frac{s}{a^2 + s^2} \right\} \sin sx \, ds.$$

i.e., $$\int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds = \frac{\pi e^{-ax}}{2} , a>0$$

Changing x by s, we get

$$\int_0^\infty \frac{x \sin sx}{x^2 + a^2} dx = \frac{\pi e^{-as}}{2} \text{------------------} (1)$$

Now $$F_s \left(\frac{x}{x^2 + a^2} \right) = \sqrt{(2/\pi)} \int_0^\infty \frac{x}{x^2 + a^2} \sin sx \, dx$$

$$= \sqrt{(2/\pi)} \frac{\pi e^{-as}}{2} , \quad \text{using (1)}$$

$$= \sqrt{(\pi/2)}\, e^{-as}$$

Similarly,for finding the Fourier cosine transform of $\frac{1}{a^2 + x^2}$, we have to find $F_c\{e^{-ax}\}$.

Consider , $F_c\{ e^{-ax} \} = \sqrt{(2/ \pi)} \int_0^\infty e^{-ax} \cos sx \; dx.$

$$= \sqrt{(2/ \pi)} \; \frac{a}{a^2 + s^2} \; .$$

Using inversion formula for Fourier cosine transforms, we get

$$e^{-ax} = \sqrt{(2/ \pi)} \int_0^\infty \left\{ \sqrt{(2/ \pi)} \; \frac{a}{a^2 + s^2} \right\} \cos sx \; ds.$$

i.e., $$\int_0^\infty \frac{\cos sx}{s^2 + a^2} \; ds = \frac{\pi e^{-ax}}{2a}$$

Changing x by s, we get

$$\int_0^\infty \frac{\cos sx}{x^2 + a^2} \; dx = \frac{\pi e^{-as}}{2a} \quad \text{-----------------------(2)}$$

Now, $$F_c\left(\frac{1}{x^2 + a^2}\right) = \sqrt{(2/ \pi)} \int_0^\infty \frac{1}{x^2 + a^2} \cos sx \; dx$$

. $$= \sqrt{(2/ \pi)} \; \frac{\pi e^{-as}}{2a} \; , \qquad \text{using (2)}$$

$$= \sqrt{(\pi/2)} \; \frac{e^{-as}}{a}$$

Example 12

Find the Fourier cosine transform of $e^{-a^2 x^2}$ and hence evaluate the Fourier sine transform of $xe^{-a^2 x^2}$.

2 2

The Fourier cosine transform of $e^{-a^2x^2}$ is given by

$$F_c\{e^{-a^2x^2}\} = \sqrt{(2/\pi)} \int_0^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \text{Real part of } \sqrt{(2/\pi)} \int_0^\infty e^{-a^2x^2} e^{isx} \, dx$$

$$= \text{Real part of } \frac{1}{a.\sqrt{2}.} e^{-s^2/4a^2}. \quad \text{(Refer example (4) of section 4.4)}$$

$$= \frac{1}{a.\sqrt{2}.} e^{-s^2/4a^2}. \text{------------------ (i)}$$

But , $F_s\{x f(x)\} = - \frac{d}{ds} F_c(s)$

$$\therefore F_s\{x e^{-a^2x^2}\} = - \frac{d}{ds}\left\{ \frac{1}{a\sqrt{2}} e^{-s^2/4a^2} \right. \quad \text{, by (1)}$$

$$= - \frac{1}{a\sqrt{2}} e^{-s^2/4a^2} (-s/2a^2).$$

$$= \frac{s}{2\sqrt{2}.a^3} e^{-s^2/4a^2}.$$

$F_c[1/\sqrt{x}] = 1/\sqrt{s}$

and $F_s[1/\sqrt{x}] = 1/\sqrt{s}$

This shows that $1/\sqrt{x}$ is self-reciprocal.

Example 13

Evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ using transform methods.

Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

Then $F_c\{ s \} = \sqrt{(2/\pi)} \int_0^{\infty} e^{-ax} \cos sx \, dx.$

$$= \sqrt{(2/\pi)} \frac{a}{a^2 + s^2} .$$

Similarly, $G_c\{ s \} = \sqrt{(2/\pi)} \dfrac{b}{b^2 + s^2}.$

Now using Parseval‟s identity for Fourier cosine transforms,

i.e., $\int_0^{\infty} F_c(s) . G_c(s) \, ds = \int_0^{\infty} f(x) \, g(x) dx.$

we have, $\dfrac{2}{\pi} \displaystyle\int_0^{\infty} \frac{ab}{(a^2 + s^2)(b^2 + s^2)} \, ds = \int_0^{\infty} e^{-(a+b)x} \, dx$

or $\dfrac{2ab}{\pi} \displaystyle\int_0^{\infty} \frac{ds}{(a^2 + s^2)(b^2 + s^2)} = \left\{ \frac{e^{-(a+b)x}}{-(a+b)} \right\}_0^{\infty}$

$$= 1 / (a+b)$$

Thus, $\displaystyle\int_0^{\infty} \frac{dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2ab(a+b)}$

Example 14

Using Parseval‟s identity, evaluate the integrals

$$\int_0^{\infty} \frac{dx}{(a^2 + x^2)^2} \quad \text{and} \quad \int_0^{\infty} \frac{x^2}{(a^2 + x^2)^2} \, dx \quad \text{if } a > 0$$

Let $f(x) = e^{-ax}$

Then $F_s(s) = \sqrt{(2/\pi)} \dfrac{s}{a^2 + s^2}$,

$F_c(s) = \sqrt{(2/\pi)} \dfrac{a}{a^2 + s^2}$

Now, Using Parseval‟s identity for sine transforms,

i.e., $$\int_0^{\infty} \left| F_s(s) \right|^2 \, ds = \int_0^{\infty} \left| f(x) \right|^2 dx .$$

we get, $$(2/\pi) \int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} \, ds = \int_0^{\infty} e^{-2ax} \, dx$$

or $$(2/\pi) \int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} \, ds = \left\{ \frac{e^{-2ax}}{-2a} \right\}_0^{\infty} = \frac{1}{2a}$$

Thus $$\int_0^{\infty} \frac{x^2}{(a^2 + x^2)^2} \, dx = \frac{\pi}{4a} \text{, if } a > 0$$

Now, Using Parseval‟s identity for cosine transforms,

i.e., $$\int_0^{\infty} \left| F_c(s) \right|^2 \, ds = \int_0^{\infty} \left| f(x) \right|^2 dx .$$

we get, $$(2/\pi) \int_0^{\infty} \frac{a^2}{(a^2 + s^2)^2} \, ds = \int_0^{\infty} e^{-2ax} \, dx$$

or $$(2a^2/\pi) \int_0^{\infty} \frac{ds}{(a^2 + s^2)^2} = \frac{1}{2a}$$

Thus , $$\int_0^{\infty} \frac{dx}{(a^2 + x^2)^2} = \frac{\pi}{4a^3} \text{, if } a > 0$$

Exercises

1. Find the Fourier sine transform of the function

$$f(x) = \begin{cases} \sin x, & 0 \le x < a. \\ 0, & x > a \end{cases}$$

2. Find the Fourier cosine transform of e^{-x} and hence deduce by using the inversion formula

$$\int_0^\infty \frac{\cos \alpha x \, dx}{(1 + x^2)} = \frac{\pi}{2} e^{-\alpha}$$

3. Find the Fourier cosine transform of $e^{-ax}\sin ax$.

4. Find the Fourier cosine transform of $e^{-2x} + 3 e^{-x}$

5. Find the Fourier cosine transform of
   (i) e^{-ax} / x (ii) $(e^{-ax} - e^{-bx}) / x$

6. Find, when $n > 0$
   (i) $F_s[x^{n-1}]$ and (ii) $F_c[x^{n-1}]$ Hint: $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$,n>0,a>0

7. Find $F_c[xe^{-ax}]$ and $F_s[xe^{-ax}]$

8. Show that the Fourier sine transform of $1 / (1 + x^2)$ is $\sqrt{(\pi/2)}\, e^{-s}$.

9. Show that the Fourier sine transform of $x / (1 + x^2)$ is $\sqrt{(\pi/2)}\, e^{-s}$.

10. Show that $x\, e^{-x^2/2}$ is self reciprocal with respect to Fourier sine transform.

11. Using transform methods to evaluate

(i) $$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$$ and

UNIT-V

Z – Transforms AND DIFFERENCE EQUATIONS

Introduction

The Z-transform plays a vital role in the field of communication Engineering and control Engineering, especially in digital signal processing. Laplace transform and Fourier transform are the most effective tools in the study of continuous time signals, where as Z – transform is used in discrete time signal analysis. The application of Z – transform in discrete analysis is similar to that of the Laplace transform in continuous systems. Moreover, Z-transform has many properties similar to those of the Laplace transform. But, the main difference is Z-transform operates only on sequences of the discrete integer-valued arguments. This chapter gives concrete ideas about Z-transforms and their properties. The last section applies Z-transforms to the solution of difference equations.

Difference Equations

Difference equations arise naturally in all situations in which sequential relation exists at various discrete values of the independent variables. These equations may be thought of as the discrete counterparts of the differential equations. Z-transform is a very useful tool to solve these equations.

A **difference equation** is a relation between the independent variable, the dependent variable and the successive differences of the dependent variable.

For example, $\Delta^2 y_n + 7\Delta y_n + 12 y_n = n^2$ ----------------------------(i)

and $\Delta^3 y_n - 3\Delta y_n - 2y_n = \cos n$ -------------------------(ii)

are difference equations.

The differences Δy_n, $\Delta^2 y_n$, etc can also be expressed as.

$$\Delta y_n = y_{n+1} - y_n,$$
$$\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n.$$
$$\Delta^3 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n \text{ and so on.}$$

Substituting these in (i) and (ii), the equations take the form

$$y_{n+2} + 5y_{n+1} + 6y_n = n^2 \quad \text{-----------} \quad (iii)$$

and $\quad y_{n+3} - 3y_{n+2} = \cos n \text{ -------------------} \; (iv)$

Note that the above equations are free of Δ's.

If a difference equation is written in the form free of Δ's, then the **order** of the difference equation is the difference between the highest and lowest subscripts of y''s occurring in it. For example, the order of equation (iii) is 2 and equation (iv) is 1.

The highest power of the y′s in a difference equation is defined as its **degree** when it is written in a form free of Δ's. For example, the degree of the equations

$y_{n+3} + 5y_{n+2} + y_n = n^2 + n + 1$ is 3 and $y^3_{n+3} + 2y_{n+1}\, y_n = 5$ is 2.

Linear Difference Equations

A **linear difference equation with constant coefficients** is of the form

$$a_0\, y_{n+r} + a_1\, y_{n+r-1} + a_2\, y_{n+r-2} + \text{.........} + a_r y_n = \phi(n).$$

i.e., $(a_0E^r + a_1E^{r-1} + a_2\, E^{r-2} + \dots + a_r)y_n = \phi(n)$ -------- (1)

where $a_0, a_1, a_2, \dots a_r$ are constants and $\phi(n)$ are known functions of n.

The equation (1) can be expressed in symbolic form as

$$f(E)\, y_n = \phi(n) \text{ ---------------------------------------}(2)$$

If $\phi(n)$ is zero, then equation (2) reduces to

$$f(E)\, y_n = 0 \text{ --------------------------------(3)}$$

which is known as the **homogeneous difference equation** corresponding to (2).The solution
of (2) consists of two parts, namely, the complementary function and the particular integral.
The solution of equation (3) which involves as many arbitrary constants as the order of the equation is called the **complementary function**. The **particular integral** is a particular solution of equation(1) and it is a function of „n‟ without any arbitrary constants.

Thus the complete solution of (1) is given by y_n = C.F + P.I.

Example 1

Form the difference equation for the Fibonacci sequence .

The integers 0,1,1,2,3,5,8,13,21,........ are said to form a Fibonacci sequence.

If y_n be the n^{th} term of this sequence, then

$$y_n = y_{n-1} + y_{n-2} \text{ for } n > 2$$

$$\text{or } y_{n+2} - y_{n+1} - y_n = 0 \text{ for } n > 0$$

Z - Transforms and its Properties

Definition

Let $\{f_n\}$ be a sequence defined for n = 0,1,2,…….,then its Z-transform F(z) is defined as

$$F(z) = Z\{f_n\} = \sum_{n=0}^{\infty} f_n z^{-n} ,$$

whenever the series converges and it depends on the sequence $\{f_n\}$.

The inverse Z-transform of F(z) is given by $Z^{-1}\{F(z)\} = \{f_n\}$.

Note: If $\{f_n\}$ is defined for n = 0, ± 1, ± 2, ..., then

$$F(z) = Z\{f_n\} = \sum_{n=-\infty}^{\infty} f_n z^{-n}$$, which is known as the two – sided Z- transform.

Properties of Z-Transforms

1. The Z-transform is linear.

i.e, if $F(z) = Z\{f_n\}$ and $G(z) = Z\{g_n\}$, then

$$Z\{af_n + bg_n\} = aF(z) + bG(z).$$

Proof:

$$Z\{ af_n + bg_n\} = \sum_{n=0}^{\infty} \{ af_n + bg_n\} z^{-n} \quad \text{(by definition)}$$

$$= a \sum_{n=0}^{\infty} f_n z^{-n} + b\sum_{n=0}^{\infty} g_n z^{-n}$$

$$= aF(z) + b\, G(z)$$

2. If $Z\{f_n\} = F(z)$, then $Z\{a^n f_n\} = F(z/a)$

Proof: By definition, we have

$$Z\{ a^n f_n\} = \sum_{n=0}^{\infty} a^n f_n z^{-n}$$

$$= \sum_{n=0}^{\infty} f_n (z/a)^{-n} = F(z/a)$$

Corollary:

If $Z\{f_n\} = F(z)$, then $Z\{ a^{-n} f_n\} = F(az)$.

3. $Z\{nf_n\} = -z \dfrac{dF(z)}{dz}$

Proof

We have $F(z)= \sum_{n=0}^{\infty} f_n z^{-n}$

Differentiating, we get

$$\frac{dF(z)}{dz} = \sum_{n=0}^{\infty} f_n (-n) z^{-n-1}$$

$$= -\frac{1}{z} \sum_{n=0}^{\infty} nf_n z^{-n}$$

$$= -\frac{1}{z} Z\{nf_n\}$$

Hence, $Z\{nf_n\} = -z \dfrac{dF(z)}{dz}$

4. If $Z\{f_n\} = F(z)$, then

$$Z\{f_{n+k}\} = z^k\{ F(z) - f_0 - (f_1 / z) - \ldots - (f_{k-1} / z^{k-1}) \} \ (k > 0)$$

Proof

$$Z\{ f_{n+k}\} = \sum_{n=0}^{\infty} f_{n+k} z^{-n} \text{, by definition.}$$

$$= \sum_{n=0}^{\infty} f_{n+k} z^{-n} z^k z^{-k}$$

$$= z^k \sum_{n=0}^{\infty} f_{n+k} z^{-(n+k)}$$

$$= z^k \sum_{m=k}^{\infty} f_m z^{-m} \text{, where } m = n+k .$$

$$= z^k \{F(z) - f_0 - (f_1/z) - \ldots\ldots\ldots\ldots(f_{k-1} / z^{k-1}) \}$$

In Particular,

(i) $Z\{f_{n+1}\} = z \{F(z) - f_0\}$

(ii) $Z\{f_{n+2}\} = z^2 \{ F(z) - f_0 - (f_1/z) \}$

Corollary

If $Z\{f_n\} = F(z)$, then $Z\{f_{n-k}\} = z^{-k} F(z)$.

(5) Initial value Theorem

If $Z \{f_n\} = F(z)$, then $f_o = \lim_{z \to \infty} F(z)$

Proof

We know that $F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots$

Taking limits as $z \to \infty$ on both sides, we get

$$\underset{z \to \infty}{\ell t}\, F(z) = f_0$$

Similarly, we can find

$$f_1 = \underset{z \to \infty}{\ell t} \{ z [F(z) - f_0]\}; \; f_2 = \underset{z \to \infty}{\ell t} \{ z^2 [F(z) - f_0 - f_1 z^{-1}]\} \text{ and so on.}$$

(6) Final value Theorem

If $Z\{f_n\} = F(z)$, then $\underset{n \to \infty}{\ell t}\, f_n = \underset{z \to 1}{\ell t}\, (z-1)\, F(z)$

Proof

By definition, we have

$$Z\,\{f_{n+1} - f_n\} = \sum_{n=0}^{\infty} \{f_{n+1} - f_n\}\, z^{-n}$$

$$Z\{f_{n+1}\} - Z\{f_n\} = \sum_{n=0}^{\infty} \{f_{n+1} - f_n\}\, z^{-n}$$

$$\text{ie, } z\,\{F(z) - f_0\} - F(z) = \sum_{n=0}^{\infty} \{f_{n+1} - f_n\}\, z^{-n}$$

$$(z-1)\, F(z) - f_0 z \quad = \sum_{n=0}^{\infty} \{f_{n+1} - f_n\}\, z^{-n}$$

Taking, limits as $z \to 1$ on both sides, we get

$$\underset{z \to 1}{\ell t}\, \{(z-1)\, F(z)\} - f_0 \quad = \quad \underset{z \to 1}{\ell t} \sum_{n=0}^{\infty} \{f_{n+1} - f_n\}\, z^{-n}$$

$$= \sum_{n=0}^{\infty} (f_{n+1} - f_n) = (f_1 - f_0) + (f_2 - f_1) + \ldots + (f_{n+1} - f_n)$$

$$= \underset{n \to \infty}{\ell t}\, f_{n+1} - f_0$$

$$\text{i.e,} \quad \underset{z \to 1}{\ell t}\, \{(z-1)\, F(z)\} - f_0 = f_\infty - f_0$$

$$\text{Hence,} \qquad f_\infty = \underset{z \to 1}{\ell t}\, [(z-1)\, F(z)]$$

$$\text{i. e,} \qquad \underset{n \to \infty}{\ell t}\, f_n = \underset{z \to 1}{\ell t}\, [(z-1)\, F(z)]$$

SOME STANDARD RESULTS

1. $Z\{a^n\} = z / (z-a)$, for $|z| > |a|$.

Proof

By definition, we have

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a/z)^n$$

$$= \frac{1}{1-(a/z)}$$

$$= z / (z-a), \text{ for } |z| > |a|$$

In particular, we have

$$Z\{1\} = z / (z-1), \text{ (taking } a = 1).$$

and $$Z\{(-1)^n\} = z / (z+1), \text{ (taking } a = -1).$$

2. $Z\{na^n\} = az/(z-a)^2$

Proof: By property, we have

$$Z\{nf_n\} = -z \frac{dF(z)}{dz}$$

$$= -z \frac{d}{dz} Z\{a^n\}$$

$$\therefore Z\{na^n\} = -z \frac{d}{dz} \frac{z}{z-a} = \frac{az}{(z-a)^2}$$

Similarly, we can prove

$$Z\{n^2a^n\} = \{az(z+a)\}/ (z-a)^3$$

(3) $Z\{n^m\} = -z \dfrac{d}{dz} Z\{n^{m-1}\}$, where m is a positive integer.

Proof

$$Z\{n^m\} = \sum_{n=0}^{\infty} n^m z^{-n}$$

$$= z \sum_{n=0}^{\infty} n^{m-1}\, n\, z^{-(n+1)} \quad \text{------------------}(1)$$

Replacing m by m-1, we get

$$Z\{n^{m-1}\} = z \sum_{n=0}^{\infty} n^{m-2}\, n\, z^{-(n+1)}$$

i.e, $$Z\{n^{m-1}\} = \sum_{n=0}^{\infty} n^{m-1} z^{-n}.$$

Differentiating with respect to z, we obtain

$$\frac{d}{dz} Z\{n^{m-1}\} = \sum_{n=0}^{\infty} n^{m-1} (-n)\, z^{-(n+1)} \quad \text{-----------}(2)$$

Using (2) in (1), we get

$Z\{n^m\} = -z \dfrac{d}{dz} Z\{n^{m-1}\}$, which is the recurrence formula.

In particular, we have

$$Z\{n\} = -z \frac{d}{dz} Z\{1\}$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$$

Similarly,

$$Z\{n^2\} = -z \frac{d}{dz} Z\{n\}$$

$$= -z \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= \frac{z(z+1)}{(z-1)^3}.$$

4. $$Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} \text{ and}$$

$$Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

We know that

$$Z\{a^n\} = z/(z-a), \text{ if } |z| > |a|$$

Letting $a = e^{i\theta}$, we have

$$Z\{e^{in\theta}\} = \frac{z}{z-e^{i\theta}} = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$Z\{\cos n\theta + i\sin n\theta\} = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$= \frac{z\{(z-\cos\theta) + i\sin\theta\}}{\{(z-\cos\theta) - i\sin\theta\}\{(z-\cos\theta) + i\sin\theta\}}$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real & imaginary parts, we get

$$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} \quad \text{and}$$

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

5 . $$Z\{r^n \cos n\theta\} = \frac{z(z - r\cos\theta)}{z^2 - 2rz\cos\theta + r^2} \text{ and}$$

$$Z\{r^n \sin n\theta\} = \frac{zr \sin\theta}{z^2 - 2rz\cos\theta + r^2} \text{ if } |z|>|r|$$

We know that

$$Z\{a^n\} = z/(z-a), \text{ if } |z|>|a|$$

Letting $a = re^{i\theta}$, we have

$$Z\{r^n e^{in\theta}\} = z/(z - re^{i\theta}) .$$

i.e, $Z\{r^n (\cos n\theta + i\sin n\theta)\} = \dfrac{z}{z - re^{i\theta}}$

$$= \frac{z}{z - r(\cos\theta + i\sin\theta)}$$

$$= \frac{z\{(z - r\cos\theta) + i\, r\sin\theta\}}{\{(z - r\cos\theta) - i\, r\sin\theta\}\{(z - r\cos\theta) + i\, r\sin\theta\}}$$

$$= \frac{z(z - r\cos\theta) + i\, rz\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + i\, rz\sin\theta}{z^2 - 2rz\cos\theta + r^2}$$

Equating the Real and Imaginary parts, we get

$$Z\{r^n \cos n\theta\} = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} \text{ and}$$

$$Z\{r^n \sin n\theta\} = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2} \text{ ; if } |z| > |r|$$

Table of Z – Transforms

	f_n	$F(z)$
1.	1	$\dfrac{z}{z-1}$

2.	$(-1)^n$	$\frac{z}{z+1}$
3.	a^n	$\frac{z}{z-a}$
4.	n	$\frac{z}{(z-1)^2}$
5.	n^2	$\frac{z^2+z}{(z-1)^3}$
6.	$n(n-1)$	$\frac{2z}{(z-1)^3}$
7.	$n^{(k)}$	$\frac{k!z}{(z-1)^{k+1}}$
8.	na^n	$\frac{az}{(z-1)^2}$
9.	$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
10.	$\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
11.	$r^n\cos n\theta$	$\frac{z(z-r\cos\theta)}{z^2-2rz\cos\theta+r^2}$
12.	$r^n\sin n\theta$	$\frac{rz\sin\theta}{z^2-2rz\cos\theta+r^2}$
13.	$\cos(n\pi/2)$	$\frac{z^2}{z^2+1}$

14.	$\sin(n\pi/2)$	$\frac{z}{z^2 + 1}$
15	t	$\frac{Tz}{(z-1)^2}$
16	t^2	$\frac{T^2 z(z + 1)}{(z-1)^3}$
17	e^{at}	$\frac{z}{z - e^{aT}}$
18	e^{-at}	$\frac{z}{z - e^{-aT}}$
19	$Z\{\cos\omega t\}$	$\frac{z (z - \cos\omega T)}{z^2 - 2z \cos\omega T +1}$
20	$Z\{\sin\omega t \}$	$\frac{z \sin\omega T}{z^2 - 2z \cos\omega T +1}$
21	$Z\{e^{-at} \cos bt\}$	$\frac{ze^{aT} (ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT} \cos bT +1}$
22	$Z\{e^{-at} \sin bt\}$	$\frac{ze^{aT} \sin bT}{z^2e^{2aT} - 2ze^{aT} \cos bT +1}$

$$2\,(z-1)^3$$

$$2\,(z-1)^3$$

Example 2

Find the Z– transform of

(i) $n(n-1)$

(ii) $n^2 + 7n + 4$

(iii) $(1/2)(n+1)(n+2)$

(i) $Z\{ n(n-1)\} = Z\{n^2\} - Z\{n\}$

$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z(z+1) - z(z-1)}{(z-1)^3}$$

$$= \frac{2z}{(z-1)^3}$$

(iii) $Z\{ n^2 + 7n + 4\} = Z\{n^2\} + 7\,Z\{n\} + 4\,Z\{1\}$

$$= \frac{z(z+1)}{(z-1)^3} + 7\,\frac{z}{(z-1)^2} + 4\,\frac{z}{z-1}$$

$$= \frac{z\{(z+1) + 7(z-1) + 4(z-1)^2\}}{(z-1)^3}$$

$$= \frac{2z(z^2-2)}{(z-1)^3}$$

(iii) $Z\left\{\frac{(n+1)(n+2)}{2}\right\} = \frac{1}{2}\{ Z\{n^2\} + 3Z\{n\} + 2Z\{1\}\}$

$$= \frac{1}{2}\left\{\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{(z-1)}\right\} \text{ f } |z| > 1$$

$$= \frac{z^3}{(z-1)^3}$$

Example 3

Find the Z- transforms of $1/n$ and $1/n(n+1)$

.

$$\text{(i) } Z\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \ldots$$

$= - \log (1 – 1/z)$ if $|1/z| < 1$

$= - \log (z-1 / z)$

$= \log (z/z-1)$, if $| z | >1$.

$$\text{(ii) } Z\,\frac{1}{n(n+1)} = Z\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} - \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \log \frac{z}{z-1} - \left(1 + \frac{1}{2z} + \frac{1}{3z^2} + \ldots\right)$$

$$= \log \frac{z}{z-1} - z\left(\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\right)$$

$$= \log \frac{z}{z-1} - z \{ - \log (1 – 1/z)\}$$

$$= \log \frac{z}{z-1} - z \log (z/z-1)$$

$= (1- z) \log \{z/(z-1)\}$

Example 4

Find the Z- transforms of

(i) cos nπ/2
(ii) sin nπ/2

$$\text{(i) } Z\{\cos n\pi/2\} = \sum_{n=0}^{\infty} \cos \frac{n\pi}{2} z^{-n}$$

$$= 1 - \frac{1}{z^2} + \frac{1}{z^4} - \ldots\ldots\ldots\ldots\ldots..$$

$$= \left\{1 + \frac{1}{z^2}\right\}^{-1}$$

$$= \left\{\frac{z^2+1}{z^2}\right\}^{-1}$$

$$= \frac{z^2}{z^2+1}, \text{ if } |z| > |$$

$$\text{(ii) } Z\{\sin n\pi/2\} = \sum_{n=0}^{\infty} \sin \frac{n\pi}{2} z^{-n}$$

$$= \frac{1}{z} - \frac{1}{z^3} + \frac{1}{z^5} - \ldots\ldots\ldots\ldots$$

$$= \frac{1}{z}\left\{1 - \frac{1}{z^2} + \frac{1}{z^4} - \ldots\right\}$$

$$= \frac{1}{z}\left\{1 + \frac{1}{z^2}\right\}^{-1}$$

$$= \frac{1}{z}\left\{\frac{z^2+1}{z^2}\right\}^{-1}$$

$$= \frac{1}{z}\,\frac{z^2}{z^2+1} = \frac{z}{z^2+1}$$

Example 5

Show that $Z\{1/n!\} = e^{1/z}$ and hence find $Z\{1/(n+1)!\}$ and $Z\{1/(n+2)!\}$

$$Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(z^{-1})^n}{n!}$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{(z^{-1})^2}{2!} + \ldots$$

$$= e^{z^{-1}} = e^{1/z}$$

To find $Z\left\{\frac{1}{(n+1)!}\right\}$

We know that $Z\{f_{n+1}\} = z\{F(z) - f_0\}$

Therefore,

$$Z\left\{\frac{1}{(n+1)!}\right\} = z\left\{Z\frac{1}{n!} - 1\right\}$$

$$= z\{e^{1/z} - 1\}$$

Similarly,

$$Z\left\{\frac{1}{(n+2)!}\right\} = z^2\{e^{1/z} - 1 - (1/z)\}.$$

Example 6

Find the Z- transforms of the following

(i) $f(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

(ii) $f(n) = \begin{cases} 0, & \text{if } n > 0 \\ 1, & \text{if } n \leq 0 \end{cases}$

(iii)f(n) = $a^n / n!, n \geq 0$
0, otherwise

(i) $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$= \sum_{n=0}^{\infty} n z^{-n}$

$= (1 / z) + (2/ z^2) + (3/ z^3) + . . .$

$= (1 / z) \{1+ (2/z) + (3/z^2) + . . .\}$

$= (1/z)\{ 1 – (1/z)\}^{-2}$

$= 1/z \left(\frac{z-1}{z}\right)^{-2}$

$= z / (z-1)^2$, if |z |> |

(ii) $Z\{f(n)\}= \sum_{n=-\infty}^{\infty} f(n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} z^{-n}$

$= \sum_{n=0}^{\infty} z^{n}$

$= (1/1 – z)$, if | z | < 1.

(iii) $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$

$= \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!}$

$= e^{az^{-1}} = e^{a/z}$

Example 7

If $F(z) = \frac{2z^2 + 3z +12}{(z-1)^4}$, find the value of „$f_2$" and „$f_3$".

$2z^2 + 3z +12$

Given that $F(z) = \dfrac{}{(z-1)^4}$.

This can be expressed as

$$F(z) = \frac{1}{z^2} \cdot \frac{2 + 3z^{-1} + 12z^{-2}}{(1- z^{-1})^4}.$$

By the initial value theorem, we have

$$f_o = \ell t_{z \to \infty} F(z) = 0.$$

Also, $f_1 = \ell t_{z \to \infty} \{z[F(z) - f_o]\} = 0.$

Now, $f_2 = \ell t_{z \to \infty} \{z^2 [F(z) - f_o – (f_1 /z)]\}$

$$= \ell t_{z \to \infty} \frac{2 + 3z^{-1} + 12z^{-2}}{(1- z^{-1})^4} - 0 – 0.$$

$= 2.$

and $f_3 = \ell t_{z \to \infty} \{z^3 [F(z) - f_o – (f_1 /z) – (f_2/ z^2)]\}$

$$= \ell t_{z \to \infty} z^3 \left[\frac{2 + 3z^{-1} + 12z^{-2}}{(1- z^{-1})^4} - \frac{2}{z^2} \right]$$

Given that $= \ell t_{z \to \infty} z^3 \dfrac{11z^3 + 8z - 2}{z^2 (z-1)^4}. = 11.$

Inverse Z – Transforms

The inverse Z – transforms can be obtained by using any one of the following methods.They are

I. Power series method
II. Partial fraction method
III. Inversion Integral method
IV. Long division method

I. Power series method

This is the simplest method of finding the inverse Z –transform. Here F(z) can be expanded in a series of ascending powers of z^{-1} and the coefficient of z^{-n} will be the desired inverse Z- transform.

Example 8

Find the inverse Z – transform of log {z /(z+1)} by power series method.

$$\text{Putting } z = \frac{1}{y},\ F(z) = \log \frac{1/y}{(1/y)+1}$$

$$= \log \frac{1}{1+y}$$

$$= -\log(1+y)$$

$$= -y + \frac{y^2}{2} - \frac{y^3}{3} + \ldots\ldots$$

$$= -z^{-1} + \frac{1}{2} z^{-2} - \frac{1}{3} z^{-3} + \ldots\ldots + \frac{(-1)^n}{n} z^{-n}$$

$$\text{Thus, } f_n = \begin{cases} 0, & \text{for } n = 0 \\ (-1)^n / n, & \text{otherwise} \end{cases}$$

II. Partial Fraction Method

Here, F(z) is resolved into partial fractions and the inverse transform can be taken directly.

Example 9

Find the inverse Z – transform of $\frac{z}{z^2 + 7z + 10}$

$$\text{Let } F(z) = \frac{z}{z^2 + 7z + 10}$$

$$\text{Then } \frac{F(z)}{z} = \frac{1}{z^2 + 7z + 10} = \frac{1}{(z+2)(z+5)}$$

$$\text{Now, consider } \frac{1}{(z+2)(z+5)} = \frac{A}{z+2} + \frac{B}{z+5}$$

$$= \frac{1}{3}\,\frac{1}{z+2} - \frac{1}{3}\,\frac{1}{z+5}$$

$$\text{Therefore, } F(z) = \frac{1}{3}\,\frac{z}{z+2} - \frac{1}{3}\,\frac{z}{z+5}$$

Inverting, we get

$$= \frac{1}{3}(-2)^n \frac{1}{3}(-5)^n$$

Example 10

Find the inverse Z – transform of $\frac{8z^2}{(2z-1)(4z-1)}$

$$\text{Let } F(z) = \frac{8z^2}{(2z-1)(4z-1)} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$\text{Then } \frac{F(z)}{z} = \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$\text{Now, } \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{4}}$$

$$\text{We get, } \frac{F(z)}{z} = \frac{2}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{4}}$$

Therefore, $F(z) = 2 \dfrac{z}{z - \frac{1}{2}} - \dfrac{z}{z - \frac{1}{4}}$

Inverting, we get

$$f_n = Z^{-1}\{F(z)\} = 2\, Z^{-1} \left[\frac{z}{z - \frac{1}{2}}\right] - Z^{-1} \left[\frac{z}{z - \frac{1}{4}}\right]$$

i.e, $f_n = 2\,(1/2)^n - (1/4)^n$, n = 0, 1, 2, ...

Example 11

Find $Z^{-1} \left\{ \dfrac{4 - 8z^{-1} + 6z^{-2}}{(1+z^{-1})\,(1-2z^{-1})^2} \right\}$ by the method of partial fractions.

Let $F(z) = \dfrac{4 - 8z^{-1} + 6z^{-2}}{(1+z^{-1})\,(1-2z^{-1})^2}$

$$= \frac{4z^3 - 8z^2 + 6z}{(z+1)\,(z-2)^2}$$

Then $\dfrac{F(z)}{z} = \dfrac{4z^2 - 8z + 6}{(z+1)\,(z-2)^2} = \dfrac{A}{z+1} + \dfrac{B}{z-2} + \dfrac{C}{(z-2)^2}$, where A = B = C = 2.

So that $\dfrac{F(z)}{z} = \dfrac{2}{z+1} + \dfrac{2}{z-2} + \dfrac{2}{(z-2)^2}$

Hence, $F(z) = \dfrac{2z}{z+1} + \dfrac{2z}{z-2} + \dfrac{2z}{(z-2)^2}$

Inverting, we get

$$f_n = 2(-1)^n + 2(2)^n + n.2^n$$

i.e, $f_n = 2(-1)^n + (n+2)2^n$

Inversion Integral Method or Residue Method

The inverse Z-transform of F (z) is given by the formula

$$f_n = \frac{1}{2\pi i} \int_C F(z)\, z^{n-1}\, dz$$

= Sum of residues of $F(z).z^{n-1}$ at the poles of F(z) inside the contour C which is drawn according to the given Region of convergence.

Example 12

Using the inversion integral method, find the inverse Z-transform of

$$\frac{3z}{(z-1)(z-2)}$$

$$\text{Let} \quad F(z) = \frac{3z}{(z-1)(z-2)}.$$

Its poles are z = 1,2 which are simple poles.
By inversion integral method, we have

$$f_n = \frac{1}{2\pi i} \int_C F(z).\, z^{n-1}\, dz = \text{sum of resides of } F(z).\, z^{n-1} \text{ at the poles of } F(z).$$

$$\text{i.e, } f_n = \frac{1}{2\pi i} \int_C \frac{3z}{(z-1)(z-2)}.\, z^{n-1}\, dz = \frac{1}{2\pi i} \int_C \frac{3z^n}{(z-1)(z-2)} dz = \text{sum of residues} \qquad \text{---------(1).}$$

Now,

$$\text{Residue (at } z = 1) = \lim_{z \to 1} (z-1) \frac{3z^n}{(z-1)(z-2)} = -3$$

$$\text{Residue (at } z = 2) = \lim_{z \to 2} (z-2) \frac{3z^n}{(z-1)(z-2)} = 3.2^n$$

$\therefore$ Sum of Residues = $-3 + 3.2^n = 3(2^n-1)$.

Thus the required inverse Z-transform is

$$f_n = 3(2^n-1),\ n = 0, 1, 2, \ldots$$

Example 13

Find the inverse z-transform of $\dfrac{z(z+1)}{(z-1)^3}$ by residue method

$$\text{Let } F(z) = \frac{z(z+1)}{(z-1)^3}$$

The pole of F(z) is z = 1, which is a pole of order 3.
By Residue method, we have

$$f_n = \frac{1}{2\pi i}\int_C F(z).\, z^{n-1}\, dz = \text{sum of residues of } F(z).z^{n-1} \text{ at the poles of } F(z)$$

$$\text{i.e., } f_n = \frac{1}{2\pi i}\int_C z^n \,.\, \frac{z+1}{(z-1)^3}\, dz = \text{sum of residues .}$$

$$\text{Now, Residue (at } z = 1) = \frac{1}{2!}\ \underset{z\to 1}{\ell t}\ \frac{d^2}{dz^2}\ (z-1)^3\ \frac{z^n(z+1)}{(z-1)^3}$$

$$= \frac{1}{2!}\ \underset{z\to 1}{\ell t}\frac{d^2}{dz^2}\{ z^n (z+1)\}$$

$$= \frac{1}{2!}\ \underset{z\to 1}{\ell t}\frac{d^2}{dz^2}\{ z^{n+1} + z^n)\}$$

$$= \frac{1}{2}\ \underset{z\to 1}{\ell t}\ \{ n(n+1)\, z^{n-1} + n(n-1)\, z^{n-2} \}$$

$$= \frac{1}{2}\{ n(n+1) + n(n-1)\} = n^2$$

Hence, $f_n = n^2$, n=0,1,2,…..

IV. Long Division Method

If F(z) is expressed as a ratio of two polynomials, namely, $F(z) = g(z^{-1}) / h(z^{-1})$, which can not be factorized, then divide the numerator by the denominator and the inverse transform can be taken term by term in the quotient.

Example 14

Find the inverse Z-transform of $\dfrac{1+2z^{-1}}{1-z^{-1}}$, by long division method

Let $F(z) = \dfrac{1+2z^{-1}}{1-z^{-1}}$

By actual division,

$$
\begin{array}{r|l}
 & 1+3z^{-1}+3z^{-2}+3z^{-3} \\
\hline
1-z^{-1} & 1+2z^{-1} \\
 & 1-z^{-1} \\
\hline
 & \quad +3z^{-1} \\
 & \quad 3z^{-1}-3z^{-2} \\
\hline
 & \qquad +3z^{-2} \\
 & \qquad 3z^{-2}-3z^{-3} \\
\hline
 & \qquad\quad +3z^{-3} \\
 & \qquad\quad 3z^{-3}-3z^{-4} \\
\hline
 & \qquad\qquad +3z^{-4}
\end{array}
$$

Thus $F(z) = 1 + 3z^{-1} + 3z^{-2} + 3z^{-3} + \ldots\ldots$

Now, Comparing the quotient with

$$\sum_{n=0}^{\infty} f_n z^{-n} = f_o + f_1 z^{-1} + f_2 z^{-2} + f_3 z^{-3} + \ldots\ldots$$

We get the sequence f_n as $f_0 = 1, f_1 = f_2 = f_3 = \ldots\ldots\ldots = 3$.

Hence $f_n = \begin{cases} 1, & \text{for } n = 0 \\ 3, & \text{for } n \geq 1 \end{cases}$

Example 15

Find the inverse Z-transform of $\dfrac{z}{z^2 - 3z + 2}$

By actual division

$$
\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 7z^{-3} + \dots \dots \dots \\
\hline
1-3z^{-1} + 2z^{-2} & z^{-1} \\
 & z^{-1} - 3z^{-2} + 2z^{-3} \\
\hline
 & \quad 3z^{-2} - 2z^{-3} \\
 & \quad 3z^{-2} - 9z^{-3} + 6z^{-4} \\
\hline
 & \qquad 7z^{-3} - 6z^{-4} \\
 & \qquad 7z^{-3} - 21z^{-4} + 14z^{-5} \\
\hline
 & \qquad +15z^{-4} - 14z^{-5} \\
\hline
\end{array}
$$

$\therefore F(z) = z^{-1} + 3z^{-2} + 7z^{-3} + \dots \dots \dots$

Now comparing the quotient with

$$\sum_{n=0}^{\infty} f_n z^{-n} = f_0 + f_1 z^{-1} + f_2 z^{-2} + f_3 z^{-3} + \dots .$$

We get the sequence f_n as $f_0 = 0, f_1 = 1, f_2 = 3, f_3 = 7, \dots \dots \dots$

Hence, $f_n = 2^n - 1, n = 0, 1, 2, 3, \dots$

Exercises

1. Find $Z^{-1}\{4z / (z-1)^3\}$ by the long division method

2. Find $Z^{-1} \dfrac{z(z^2 - z + 2)}{(z+1)(z-1)^2}$ by using Residue theorem

3. Find $Z^{-1} \dfrac{z^2}{(z+2)(z^2+4)}$ by using Residue theorem

4. Find $Z^{-1}(z/z-a)$ by power series method

5. Find $Z^{-1}(e^{-2/z})$ by power series method

6. Find $Z^{-1}\left[\dfrac{z^3 - 20z}{(z-4)(z-2)^3}\right]$ by using Partial fraction method

CONVOLUTION THEOREM

If $Z^{-1}\{F(z)\} = f_n$ and $Z^{-1}\{G(z)\} = g_n$, then

$$Z^{-1}\{F(z).\,G(z)\} = \sum_{m=0}^{n} f_m .\, g_{n-m} = f_n * g_n,$$ where the symbol * denotes the operation of convolution.

Proof

We have $F(z) = \sum_{n=0}^{\infty} f_n z^{-n}$, $G(z) = \sum_{n=0}^{\infty} g_n z^{-n}$

$\therefore F(z).G(z) = (f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots + f_n z^{-n} + \ldots \infty).\,(g_o + g_1 z^{-1} + g_2 z^{-2} + \ldots + g_n z^{-n} + \ldots \infty)$

$$= \sum_{n=0}^{\infty} (f_o g_n + f_1 g_{n-1} + f_2 g_{n-2} + \ldots + f_n g_o) z^{-n}$$

$$= Z\,(f_o g_n + f_1 g_{n-1} + f_2 g_{n-2} + \ldots + f_n g_o)$$

$$= Z \sum_{m=0}^{n} f_m\, g_{n-m}$$

$$= Z\,\{f_n * g_n\}$$

Hence, $Z^{-1}\{F(z) . G(z)\} = f_n * g_n$

Example 16

Use convolution theorem to evaluate

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$$

We know that $Z^{-1}\{F(z).\,G(z)\} = f_n * g_n$.

Let $F(z) = \dfrac{z}{z-a}$ and $G(z)\ \dfrac{z}{z-b}$

Then $f_n = Z^{-1}\left(\dfrac{z}{z-a}\right) = a^n$ & $g_n = Z^{-1}\left(\dfrac{z}{z-b}\right) = b^n$

Now,

$$Z^{-1}\{F(z).\,G(z)\} = f_n * g_n = a^n * b^n$$

$$= \sum_{m=0}^{n} a^m b^{n-m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m \text{ which is a G.P.}$$

$$= b^n \frac{(a/b)^{n+1} - 1}{(a/b) - 1}$$

$$\text{ie, } Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = \frac{a^{n+1} - b^{n+1}}{a-b}$$

Example 17

Find $z^{-1}\left\{\left(\frac{z}{(z-1)}\right)^3\right\}$ by using convolution theorem

Let $F(z) = \frac{z^2}{(z-1)^2}$ and $G(z) = \frac{z}{(z-1)}$

Then $f_n = n+1$ & $g_n = 1$

By convolution Theorem, we have

$$Z^{-1}\{F(z).\,G(z)\} = f_n * g_n = (n+1) * 1 = \sum_{m=0}^{n} (m+1)\,.\,1$$

$$= \frac{(n+1)(n+2)}{2}$$

Example 18

Use convolution theorem to find the inverse Z- transform of

$$\frac{1}{[1-(1/2)z^{-1}]\,[1-(1/4)z^{-1}]}$$

Given $Z^{-1}\left\{\dfrac{1}{[1-(1/2)z^{-1}]\,[1-(1/4)z^{-1}]}\right\} = Z^{-1}\left\{\dfrac{z^2}{[z-(1/2)]\,[z-(1/4)]}\right\}$

Let $F(z) = \dfrac{z}{z-(1/2)}$ & $G(z) = \dfrac{z}{z-(1/4)}$

Then $f_n = (1/2)^n$ & $g_n = (1/4)^n$.

We know that $Z^{-1}\{F(z).\,G(z)\} = f_n * g_n$

$$= (1/2)^n * (1/4)^n$$

$$= \sum_{m=0}^{n} \left(\frac{1}{2}\right)^{m} \left(\frac{1}{4}\right)^{n-m}$$

$$= \left(\frac{1}{4}\right)^{n} \sum_{m=0}^{n} \left(\frac{1}{2}\right)^{m} \left(\frac{1}{4}\right)^{-m}$$

$$= \left(\frac{1}{4}\right)^{n} \sum_{m=0}^{n} 2^{m}$$

$$= \left(\frac{1}{4}\right)^{n} \{1+2+2^2+ \ldots\ldots + 2^n\} \text{ which is a G.P}$$

$$= \left(\frac{1}{4}\right)^{n} \frac{2^{n+1}-1}{2-1}$$

$$= \left(\frac{1}{4}\right)^{n} \{2^{n+1}-1\}$$

$$= \frac{1}{2^{n-1}} - \left(\frac{1}{4}\right)^{n}$$

$$\therefore Z^{-1}\left\{\frac{1}{[1-(1/2)z^{-1}]\,[1-(1/4)z^{-1}]}\right\} = \frac{1}{2^{n-1}} - \frac{1}{4^{n}}$$

Application of Z - transform to Difference equations

As we know, the Laplace transforms method is quite effective in solving linear differential equations, the Z - transform is useful tool in solving linear difference equations.

To solve a difference equation, we have to take the Z - transform of both sides of the difference equation using the property

$$Z\{f_{n+k}\} = z^k\{ F(z) – f_0 – (f_1 / z) - \dots - (f_{k-1} / z^{k-1}) \} \ (k > 0)$$

Using the initial conditions, we get an algebraic equation of the form $F(z) = \phi(z)$.

By taking the inverse Z-transform, we get the required solution f_n of the given difference equation.

Exmaple 19

Solve the difference equation $y_{n+1} + y_n = 1$, $y_0 = 0$, by Z - transform method.

Given equation is $y_{n+1} + y_n = 1$--(1)

Let Y(z) be the Z -transform of $\{y_n\}$.

Taking the Z - transforms of both sides of (1), we get

$$Z\{y_{n+1}\} + Z\{y_n\} = Z\{1\}.$$

ie, $z \{Y(z) - y_0\} + Y(z) = z /(z-1)$.

Using the given condition, it reduces to

$$(z+1)\, Y(z) = \frac{z}{z - 1}$$

z

$$\text{i.e, } Y(z) = \frac{\quad}{(z - 1)\ (z + 1)}$$

$$\text{or} \qquad Y(z) = \frac{1}{2}\left[\frac{z}{z - 1} - \frac{z}{z + 1}\right]$$

On taking inverse Z-transforms, we obtain

$$y_n = (1/2)\{1 - (-1)^n\}$$

Example 20

Solve $y_{n+2} + y_n = 1$, $y_0 = y_1 = 0$, using Z-transforms.

Consider $\quad y_{n+2} + y_n = 1$---------------- (1)

Taking Z- transforms on both sides, we get

$$Z\{y_{n+2}\} + Z\{y_n\} = Z\{1\}$$

$$z^2 \left\{Y(z) - y_0 - \frac{y_1}{z}\right\} + Y(z) = \frac{z}{z - 1}$$

$$(z^2 + 1)\ Y(z) = \frac{z}{z - 1}$$

$$\text{or } Y(z) = \frac{z}{(z - 1)\ (z^2 + 1)}$$

$$\text{Now,} \qquad \frac{Y(z)}{z} = \frac{1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz + C}{z^2+1}$$

$$= \frac{1}{2}\left[\frac{1}{z - 1} - \frac{z}{z^2 + 1} - \frac{1}{z^2 + 1}\right]$$

$$\text{Therefore,} \qquad Y(z) = \frac{1}{2}\left[\frac{z}{z - 1} - \frac{z^2}{z2 + 1} - \frac{z}{z2 + 1}\right]$$

Using Inverse Z-transform, we get

$$y_n = (½)\{1 - \cos(n\pi / 2) - \sin(n\pi / 2)\}.$$

Example 21

Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, $y_0 = y_1 = 0$, using Z-transforms.

Consider $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ ----------- (1)

Taking the Z-transform of both sides, we get

$$Z\{y_{n+2}\} + 6Z\{y_{n+1}\} + 9Z\{y_n\} = Z\{2^n\}$$

i.e, $$z^2 \; Y(z) - y_0 - \frac{y_1}{z} + 6z\{Y(z) - y_0\} + 9Y(z) = \frac{z}{z-2}$$

$$(z^2 + 6z + 9)\,Y(z) = \frac{z}{z-2}$$

i.e, $$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

Therefore, $$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

ie, $$\frac{Y(z)}{z} = \frac{1}{25}\,\frac{1}{z-2} \quad \frac{1}{25}\,\frac{1}{z+3} \quad \frac{1}{5}\,\frac{1}{(z+3)^2},$$

using partial fractions.

Or $$Y(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} - \frac{5z}{(z+3)^2}\right]$$

On taking Inverse Z-transforms, we get

$$y_n = (1/25)\{2^n - (-3)^n + (5/3)\, n\, (-3)^n\}.$$

Example 22

Solve the simultaneous equations

$$x_{n+1} - y_n = 1;\ y_{n+1} - x_n = 1 \text{ with } x(0) = 0;\ y(0) = 0.$$

The given equations are

$$x_{n+1} - y_n = 1, \qquad x_0 = 0 \qquad \text{-------------- (1)}$$
$$y_{n+1} - x_n = 1, \qquad y_0 = 0 \qquad \text{-------------- (2)}$$

Taking Z-transforms, we get

$$z\{X(z) - x_0\} - Y(z) = \frac{z}{z-1}$$

$$z\{Y(z) - y_0\} - X(z) = \frac{z}{z-1}$$

Using the initial conditions, we have

$$z\,X(z) - Y(z) = \frac{z}{z-1}$$

$$z\,Y(z) - X(z) = \frac{z}{z-1}$$

Solving the above equations, we get

$$X(z) = \frac{z}{(z-1)^2} \quad \text{and} \quad Y(z) = \frac{z}{(z-1)^2}.$$

On taking the inverse Z-transform of both sides, we have $x_n = n$ and $y_n = n$, which is the required solution of the simultaneous difference equations.

Example 23

Solve $x_{n+1} = 7x_n + 10y_n$; $y_{n+1} = x_n + 4y_n$, with $x_0 = 3$, $y_0 = 2$

Given

$$x_{n+1} = 7x_n + 10y_n \text{ ---------------------(1)}$$
$$y_{n+1} = x_n + 4y_n \text{ -----------------------(2)}$$

Taking Z- transforms of equation(1), we get

$$z \{ X(z) - x_0\} = 7 X(z) + 10 Y(z)$$

$$(z - 7) X(z) – 10 Y(z) = 3z \text{-----------} (3)$$

Again taking Z- transforms of equation(2), we get

$$z \{Y(z) - y_0\} = X(z) + 4Y(z)$$

$$-X(z) + (z - 4)Y(z) = 2z \text{----------} (4)$$

Eliminating „x‟ from (3) & (4), we get

$$Y(z) = \frac{2z^2 - 11z}{z^2 - 11z+8} = \frac{2z^2 - 11z}{(z-9)(z-2)}$$

so that $$\frac{Y(z)}{z} = \frac{2z - 11}{(z-9)(z-2)} = \frac{A}{z-9} + \frac{B}{z-2}$$,where A =1 and B = 1.

$$\text{ie, } \frac{Y(z)}{z} = \frac{1}{z-9} + \frac{1}{z-2}$$

$$\text{ie, } Y(z) = \frac{z}{z-9} + \frac{z}{z-2}$$

Taking Inverse Z-transforms, we get $y_n = 9^n + 2^n$.

From (2), $x_n = y_{n+1} - 4y_n = 9^{n+1} + 2^{n+1} - 4 (9^n + 2^n)$

$$= 9.9^n + 2.2^n - 4.9^n - 4.2^n$$

Therfore, $x_n = 5.9^n - 2.2^n$

Hence the solution is $x_n = 5.9^n - 2.2^n$ and $y_n = 9^n + 2^n$.

Exercises

Solve the following difference equations by Z – transform method

1. $y_{n+2} + 2y_{n+1} + y_n = n$, $y_0 = y_1 = 0$

2. $y_{n+2} - y_n = 2^n$, $y_0 = 0$, $y_1 = 1$

3. $u_{n+2} - 2\cos\alpha\, u_{n+1} + u_n = 0, u_0 = 1, u_1 =$

$\cos\alpha$ 4. $u_{n+2} = u_{n+1} + u_n$, $u_0 = 0$, $u_1 = 1$

5. $y_{n+2} - 5y_{n+1} + 6y_n = n\,(n-1)$, $y_0 = 0$, $y_1 = 0$

6. $y_{n+3} - 6y_{n+2} + 12y_{n+1} - 8y_n = 0$, $y_0 = -1$, $y_1 = 0$, $y_2 = 1$

FORMATION OF DIFFERENCE EQUATIONS

Example

Form the difference equation

$$y_n = a2^n + b(-2)^n$$

$$y_{n+1} = a2^{n+1} + b(-2)^{n+1}$$

$$= 2a2^n - 2b(-2)^n$$

$$y_{n+2} = a2^{n+2} + b(-2)^{n+1}$$

$$= 4a2^n + 4b(-2)^n$$

Eliminating a and b weget,

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -2 \\ y_{n+2} & 4 & 4 \end{vmatrix} = 0$$

$$y_n(8+8) - 1(4y_{n+1} + 2y_{n+2}) + 1(4y_{n+1} - 2y_{n+2}) = 0$$

$16y_n - 4y_{n+2} = 0$

$-4(y_{n+2} - 4y_n) = 0$

$y_{n+2} - 4y_n = 0$

Exercise:

1. Derive the difference equation form $y_n = (A + Bn)(-3)^n$

2. Derive the difference equation form $U_n = A2^n + Bn$

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2. Engineering Mathematics – Vol III – P. Kandasamy
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