PRATHYUSHA ENGINEERING COLLEGE

LECTURE NOTES

Course code	**: MA8402**
Name of the course	**: Probability and Queueing Theory**
Regulation	**: 2017**
Course faculty	**: Mr. R. Vinodkumar/AP**

MA 8402 PROBABILITY AND QUEUEING THEORY L T P C 3 1 0 4

OBJECTIVE:
To provide the required mathematical support in real life problems and develop probabilistic models which can be used in several areas of science and engineering.

UNIT I RANDOM VARIABLES 9+3
Discrete and continuous random variables – Moments – Moment generating functions – Binomia l ,Poisson, Geometric, Uniform, Exponential, Gamma and Normal distributions.

UNIT II TWO - DIMENSIONAL RANDOM VARIABLES 9+3

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and Linear regression – Transformation of random variables.

UNIT III RANDOM PROCESSES 9+3
Classification – Stationary process – Markov process - Poisson process – Discrete parameter Markov chain – Chapman Kolmogorov equations – Limiting distributions
.
UNIT IV QUEUEING MODELS 9+3
Markovian queues – Birth and Death processes – Single and multiple server queueing models – Little’s formula - Queues with finite waiting rooms – Queues with impatient customers: Balking and reneging.
UNIT V ADVANCED QUEUEING MODELS 9+3
Finite source models - M/G/1 queue – Pollaczek Khinchin formula - M/D/1 and M/E_K/1 as special cases – Series queues – Open Jackson networks.

TOTAL (L:45+T:15): 60 PERIODS OUTCOMES:

1. The students will have a fundamental knowledge of the probability concepts.

2. Acquire skills in analyzing queueing models.

3. It also helps to understand and characterize phenomenon which evolve with respect to time in a probabilistic manner.

TEXT BOOKS:
1. Ibe. O.C., "Fundamentals of Applied Probability and Random Processes", Elsevier, 1st Indian Reprint, 2007.
2. Gross. D. and Harris. C.M., "Fundamentals of Queueing Theory", Wiley Student edition, 2004.
REFERENCES: 1. Robertazzi, "Computer Networks and Systems: Queueing Theory and performance evaluation",
Springer, 3rd Edition, 2006.
2. Taha. H.A., "Operations Research", Pearson Education, Asia, 8th Edition, 2007.
3. Trivedi.K.S., "Probability and Statistics with Reliability, Queueing and Computer Science Applications", John Wiley and Sons, 2nd Edition, 2002.
4. Hwei Hsu, "Schaum’s Outline of Theory and Problems of Probability, Random Variables and Random Processes", Tata McGraw Hill Edition, New Delhi, 2004.
5. Yates. R.D. and Goodman. D. J., "Probability and Stochastic Processes", Wiley India Pvt. Ltd.,Bangalore, 2nd Edition, 2012.

UNIT - I

RANDOM VARIABLES

Introduction

Consider an experiment of throwing a coin twice. The outcomes {HH, HT, TH, TT} consider the sample space. Each of these outcome can be associated with a number by specifying a rule of association with a number by specifying a rule of association (eg. The number of heads). Such a rule of association is called a random variable. We denote a random variable by the capital letter (X, Y, etc) and any particular value of the random variable by x and y.

Thus a random variable X can be considered as a function that maps all elements in the sample space S into points on the real line. The notation X(S)=x means that x is the value associated with the outcomes S by the Random variable X.

1.1 SAMPLE SPACE

Consider an experiment of throwing a coin twice. The outcomes S = {HH, HT, TH, TT} constitute the sample space.

1.2 RANDOM VARIABLE

In this sample space each of these outcomes can be associated with a number by specifying a rule of association. Such a rule of association is called a random variables.

Eg : Number of heads

We denote random variable by the letter (X, Y, etc) and any particular value of the random variable by x or y.

S = {HH, HT, TH, TT}

X(S) = {2, 1, 1, 0}

Thus a random X can be the considered as a fun. That maps all elements in the sample space S into points on the real line. The notation X(S) = x means that x is the value associated with outcome s by the R.V.X.

Example 1.1

In the experiment of throwing a coin twice the sample space S is S = {HH,HT,TH,TT}. Let X be a random variable chosen such that X(S) = x (the number of heads).

Note

Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.

1.1.1 DISCRETE RANDOM VARIABLE

Definition : A discrete random variable is a R.V.X whose possible values consitute finite set of values or countably infinite set of values.

Example 1.1

All the R.V.'s from Example : 1 are discrete R.V's

Remark

The meaning of $P(X \leq a)$.

$P(X \leq a)$ is simply the probability of the set of outcomes 'S' in the sample space for which $X(s) \leq a$.

Or$P(X \leq a) = P\{S : X(S) \leq a\}$

In the above example : 1 we should write

$$P(X \leq 1) = P(HH, HT, TH) = \frac{3}{4}$$

Here $P(X \leq 1) = \frac{3}{4}$ means the probability of the R.V.X (the number of heads) is less than or equal to 1 is $\frac{3}{4}$.

Distribution function of the random variable X or cumulative distribution of the random variable X

Def :

The distribution function of a random variable X defined in $(-\infty, \infty)$ is given by

$$F(x) = P(X \leq x) = P\{s : X(s) \leq x\}$$

Note

Let the random variable X takes values $x_1, x_2, \ldots, x_n$ with probabilities $P_1, P_2, \ldots, P_n$ and let $x_1 < x_2 < \ldots < x_n$

Then we have

$F(x) \quad = P(X < x_1) = 0, -\infty < x < x,$

$F(x) \quad = P(X < x_1) = 0, P(X < x_1) + P(X = x_1) \qquad = 0 + p_1 = p_1$

$F(x) \quad = P(X < x_2) = 0, P(X < x_1) + P(X = x_1) + P(X = x_2) = p_1 + p_2$

$F(x) \quad = P(X < x_n) = P(X < x_1) + P(X = x_1) + \ldots.. + P(X = x_n)$

$= p_1 + p_2 + \ldots\ldots\ldots + p_n \qquad = 1$

1.1.2 PROPERTIES OF DISTRIBUTION FUNCTIONS

Property : 1 $\quad P(a < X \le b) \quad = F(b) - F(a)$, where $F(x) = P(X \le x)$

Property : 2 $\quad P(a \le X \le b) \quad = P(X = a) + F(b) - F(a)$

Property : 3 $\quad P(a < X < b) \quad = P(a < X \le b) - P(X = b)$

$= F(b) - F(a) - P(X = b)$ by prob (1)

1.1.3 PROBABILITY MASS FUNCTION (OR) PROBABILITY FUNCTION

Let X be a one dimenstional discrete R.V. which takes the values $x_1, x_2, \ldots\ldots$ To each possible outcome 'x_i' we can associate a number p_i.

i.e., $P(X = x_i) = P(x_i) = p_i$ called the probability of x_i. The number $p_i = P(x_i)$ satisfies the following conditions.

(i) $p(x_i) \ge 0, \forall_i$ (ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

The function p(x) satisfying the above two conditions is called the probability mass function (or) probability distribution of the R.V.X. The probability distribution $\{x_i, p_i\}$ can be displayed in the form of table as shown below.

$X = x_i$	x_1	x_2	…….	x_i
$P(X = x_i) = p_i$	p_1	p_2	…….	p_i

Notation

Let 'S' be a sample space. The set of all outcomes 'S' in S such that X(S) = x is denoted by writing X = x.

$P(X = x) \quad = P\{S : X(s) = x\}$

|||ly $\quad P(x \le a) \quad = P\{S : X() \in (-\infty, a)\}$

and $\quad P(a < x \le b) \quad = P\{s : X(s) \in (a, b)\}$

$P(X = a \text{ or } X = b) \quad = P\{(X = a) \cup (X = b)\}$

$P(X = a \text{ and } X = b) \quad = P\{(X = a) \cap (X = b)\}$ and so on.

Theorem 1 If X1 and X2 are random variable and K is a constant then KX_1, $X_1 + X_2$, X_1X_2, $K_1X_1 + K_2X_2$, X_1-X_2 are also random variables.

Theorem 2

If 'X' is a random variable and f(•) is a continuous function, then f(X) is a random variable.

Note

If F(x) is the distribution function of one dimensional random variable then

I. $0 \le F(x) \le 1$
II. If $x < y$, then $F(x) \le F(y)$
III. $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$
IV. $F(\infty) = \lim_{x \to \infty} F(x) = 1$
V. If 'X' is a discrete R.V. taking values x_1, x_2, x_3
Where $x_1 < x_2 < x_{i-1}\ x_i$ …….. then
$P(X = x_i) = F(x_i) - F(x_{i-1})$

Example 1.2

A random variable X has the following probability function

Values of X	0	1	2	3	4	5	6	7	8
Probability P(X)	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Determine the value of 'a'
(ii) Find $P(X<3)$, $P(X\ge3)$, $P(0<X<5)$
(iii) Find the distribution function of X.

Solution

Values of X	0	1	2	3	4	5	6	7	8
p(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) We know that if p(x) is the probability of mass function then

$$\sum_{i=0}^{8} p(x_i) = 1$$

p(0) + p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + p(8) = 1
a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1
81 a = 1
a = 1/81
put a = 1/81 in table 1, e get table 2

Table 2

X = x	0	1	2	3	4	5	6	7	8
P(x)	1/81	3/81	5/81	7/81	9/81	11/81	13/81	15/81	17/81

(ii) P(X < 3) = p(0) + p(1) + p(2)
= 1/81+ 3/81 + 5/81 = 9/81
(ii) P(X ≥ 3) = 1 - p(X < 3)
= 1 - 9/81 = 72/81
(iii) P(0 < x < 5) = p(1) + p(2) + p(3) + p(4) here 0 & 5 are not include
= 3/81 + 5/81 + 7/81 + 9/81

$$= \frac{3 + 5 + 7 + 8 + 9}{81} = \frac{24}{81}$$

(iv) To find the distribution function of X using table 2, we get

X = x	**F(X) = P(x ≤ x)**
0	F(0) = p(0) = 1/81
1	F(1) = P(X ≤ 1) = p(0) + p(1) = 1/81 + 3/81 = 4/81
2	F(2) = P(X ≤ 2) = p(0) + p(1) + p(2) = 4/81 + 5/81 = 9/81
3	F(3) = P(X ≤ 3) = p(0) + p(1) + p(2) + p(3) = 9/81 + 7/81 = 16/81
4	F(4) = P(X ≤ 4) = p(0) + p(1) + …. + p(4) = 16/81 + 9/81 = 25/81
5	F(5) = P(X ≤ 5) = p(0) + p(1) + ….. + p(4) + p(5) = 2/81 + 11/81 = 36/81
6	F(6) = P(X ≤ 6) = p(0) + p(1) + ….. + p(6) = 36/81 + 13/81 = 49/81
7	F(7) = P(X ≤ 7) = p(0) + p(1) + …. + p(6) + p(7) = 49/81 + 15/81 = 64/81

8	F(8) $= P(X \le 8) = p(0) + p(1) + \ldots.. + p(6) + p(7) + p(8)$ $= 64/81 + 17/81 = 81/81 = 1$

1.2 CONTINUOUS RANDOM VARIABLE

Def : A R.V.'X' which takes all possible values in a given internal is called a continuous random variable.

Example : Age, height, weight are continuous R.V.'s.

1.2.1 PROBABILITY DENSITY FUNCTION

Consider a continuous R.V. 'X' specified on a certain interval (a, b) (which can also be a infinite interval $(-\infty, \infty)$).

If there is a function y = f(x) such that

$$\lim_{\Delta x \to 0} \frac{P(x < X < x + \Delta x)}{\Delta x} = f(x)$$

Then this function f(x) is termed as the probability density function (or) simply density function of the R.V. 'X'.

It is also called the frequency function, distribution density or the probability density function.

The curve y = f(x) is called the probability curve of the distribution curve.

Remark :

If f(x) is p.d.f of the R.V.X then the probability that a value of the R.V. X will fall in some interval (a, b) is equal to the definite integral of the function f(x) a to b.

$$P(a < x < b) = \int_a^b f(x)\,dx$$

(or)

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

1.2.3 PROPERTIES OF P.D.F

The p.d.f f(x) of a R.V.X has the following properties

(i) $f(x) \ge 0, -\infty < x < \infty$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Remark

1. In the case of discrete R.V. the probability at a point say at x = c is not zero. But in the case of a continuous R.V.X the probability at a point is always zero.

$$P(X = c) = \int_{-\infty}^{\infty} f(x)\,dx = [x]_c^C = C - C = 0$$

2. If x is a continuous R.V. then we have $p(a \le X \le b) = p(a \le X < b)$ $= p(a < X V b)$

IMPORTANT DEFINITIONS INTERMS OF P.D.F

If f(x) is the p.d.f of a random variable ‘X’ which is defined in the interval (a, b) then

i	Arithmetic mean	$\int_a^b x\, f(x)\,dx$
ii	Harmonic mean	$\int_a^b \frac{1}{x}\, f(x)\,dx$
iii	Geometric mean ‘G’ log G	$\int_a^b \log x\, f(x)\,dx$
iv	Moments about origin	$\int_a^b x^r\, f(x)\,dx$
v	Moments about any point A	$\int_a^b (x-A)^r\, f(x)\,dx$
vi	Moment about mean μ_r	$\int_a^b (x-\text{mean})^r\, f(x)\,dx$
vii	Variance μ_2	$\int_a^b (x-\text{mean})^2\, f(x)\,dx$
viii	Mean deviation about the mean is M.D.	$\int_a^b \lvert x-\text{mean}\rvert\, f(x)\,dx$

1.2.4 Mathematical Expectations

Def : Let ‘X’ be a continuous random variable with probability density function f(x). Then the mathematical expectation of ‘X’ is denoted by E(X) and is given by

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

It is denoted by

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

Thus

$\mu_1' = E(X)$ (μ_1' about origin)

$\mu_2' = E(X^2)$ (μ_2' about origin)

$\therefore$ Mean $= \overline{X} = \mu_1' = E(X)$

And

Variance $= \mu_2' - \mu_2'^2$

Variance $= E(X^2) - [E(X)]^2$ (a)

* r^{th} moment (abut mean)

Now

$$E\{X - E(X)\}^r = \int_{-\infty}^{\infty} \{x - E(X)\}^r f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \{x - \overline{X}\}^r f(x)\,dx$$

Thus

$$\mu_r = \int_{-\infty}^{\infty} \{x - \overline{X}\}^r f(x)\,dx \qquad \text{(b)}$$

$$\text{Where} \quad \mu_r = E[X - E(X)^r]$$

This gives the r^th^ moment about mean and it is denoted by μ_r

Put r = 1 in (B) we get

$$\mu_r = \int_{-\infty}^{\infty} \{x - \overline{X}\} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x\,f(x)\,dx - \int_{-\infty}^{\infty} \overline{x}\,f(x)\,dx$$

$$= \overline{X} - \overline{X}\int_{-\infty}^{\infty} f(x)\,dx \qquad \left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

$$= \overline{X} - \overline{X}$$

$$\mu_1 = 0$$

Put r = 2 in (B), we get

$$\mu_2 = \int_{-\infty}^{\infty} (x - \overline{X})^2 f(x)\,dx$$

$$\text{Variance} = \mu_2 = E[X - E(X)]^2$$

Which gives the variance interms of expectations.

Note:

Let g(x) = K (Constant), then

$$E[g(X)] = E(K) = \int_{-\infty}^{\infty} K\,f(x)\,dx$$

$$= K\int_{-\infty}^{\infty} f(x)\,dx \qquad \left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

$$= K\,.\,1 = K$$

Thus E(K) = K $\Rightarrow$ E[a constant] = constant.

1.2.4 EXPECTATIONS (Discrete R.V.'s)

Let 'X' be a discrete random variable with P.M.F p(x)

Then

$$E(X) = \sum_x x\,p(x)$$

For discrete random variables 'X'

$$E(X^r) = \sum_x x^r \; p(x) \qquad \text{(by def)}$$

If we denote

$$E(X^r) = \mu_r'$$

Then

$$\mu_r' = E[X^r] = \sum_x x^r p(x)$$

Put r = 1, we get

$$\text{Mean } \mu_r' = \sum x \, p(x)$$

Put r = 2, we get

$$\mu_2' = E[X^2] = \sum_x x^2 p(x)$$

$$\therefore \mu_2 = \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$$

The r^{th} moment about mean

$$\mu_r' = E[\{X - E(X)\}^r]$$

$$= \sum_x (x - \overline{X})^r p(x), \qquad E(X) = \overline{X}$$

Put r = 2, we get

Variance = $\mu_2 = \sum_x (x - \overline{X})^2 \, p(x)$

ADDITION THEOREM (EXPECTATION)

Theorem :1

If X and Y are two continuous random variable with pdf $f_x(x)$ and $f_y(y)$ then
E(X+Y) = E(X) + E(Y)

*** MULTIPLICATION THEOREM OF EXPECTATION**

Theorem :2

If X and Y are independent random variables,
Then E(XY) = E(X) . E(Y)

Note :

If $X_1, X_2, \ldots\ldots, X_n$ are 'n' independent random variables, then
$E[X_1, X_2, \ldots\ldots, X_n] = E(X_1), E(X_2), \ldots\ldots, E(X_n)$

Theorem : 3

If 'X' is a random variable with pdf f(x) and 'a' is a constant, then

(i) E[a G(x)] = a E[G(x)]

(ii) E[G(x)+a] = E[G(x)+a]

Where G(X) is a function of 'X' which is also a random variable.

Theorem :4

If 'X' is a random variable with p.d.f. f(x) and 'a' and 'b' are constants, then
E[ax + b] = a E(X) + b

Cor 1:

If we take a = 1 and b = –E(X) = $-\overline{X}$, then we get

$E(X-\overline{X}) = E(X) - E(X) = 0$

Note

$E\left(\frac{1}{X}\right) \neq \frac{1}{E(X)}$

$E[\log (x)] \neq \log E(X)$

$E(X^2) \neq [E(X)]^2$

1.2.4 EXPECTATION OF A LINEAR COMBINATION OF RANDOM VARIABLES

Let $X_1, X_2, \ldots\ldots, X_n$ be any 'n' random variable and if $a_1, a_2, \ldots\ldots, a_n$ are constants, then

$E[a_1X_1 + a_2X_2 + \ldots\ldots + a_nX_n] = a_1E(X_1) + a_2E(X_2) + \ldots\ldots + a_nE(X_n)$

Result

If X is a random variable, then

$Var(aX + b) = a^2Var(X)$ 'a' and 'b' are constants.

Covariance :

If X and Y are random variables, then covariance between them is defined as

Cov(X, Y) = E{[X - E(X)] [Y - E(Y)]}

= E{XY - XE(Y) – E(X)Y + E(X)E(Y)}

Cov(X, Y) = E(XY) – E(X) . E(Y) (A)

If X and Y are independent, then

E(XY) = E(X) E(Y)

Sub (B) in (A), we get

Cov (X, Y) = 0

$\therefore$ If X and Y are independent, then

Cov (X, Y) = 0

Note

(i) Cov(aX, bY) = ab Cov(X, Y)

(ii) Cov(X+a, Y+b) = Cov(X, Y)

(iii) Cov(aX+b, cY+d) = ac Cov(X, Y)

(iv) $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\,Cov(X_1, X_2)$

If X_1, X_2 are independent

$Var(X_1 \pm X_2) = Var(X_1) + Var(X_2)$

EXPECTATION TABLE

Discrete R.V's	**Continuous R.V's**
1. $E(X) = \sum x\, p(x)$	1. $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$
2. $E(X^r) = \mu_r' = \sum_x x^r p(x)$	2. $E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r\, f(x)\, dx$

3. Mean = $\mu_r' = \sum x\, p(x)$	3. Mean = $\mu_r' = \int_{-\infty}^{\infty} x\, f(x)\, dx$
4. $\mu_2' = \sum x^2\, p(x)$	4. $\mu_2' = \int_{-\infty}^{\infty} x^2\, f(x)\, dx$
5. Variance = $\mu_2' - \mu_1'^2$ = $E(X^2) - \{E(X)\}^2$	5. Variance = $\mu_2' - \mu_1'^2$ = $E(X^2) - \{E(X)\}^2$

SOLVED PROBLEMS ON DISCRETE R.V'S

Example :1

When die is thrown, 'X' denotes the number that turns up. Find E(X), $E(X^2)$ and Var (X).

Solution

Let 'X' be the R.V. denoting the number that turns up in a die.

'X' takes values 1, 2, 3, 4, 5, 6 and with probability 1/6 for each

X = x	1	2	3	4	5	6
p(x)	1/6 $p(x_1)$	1/6 $p(x_2)$	1/6 $p(x_3)$	1/6 $p(x_4)$	1/6 $p(x_5)$	1/6 $p(x_6)$

Now

$$E(X) = \sum_{i=1}^{6} x_i\, p(x_i)$$

= $x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + x_4p(x_4) + x_5p(x_5) + x_6p(x_6)$

= 1 x (1/6) + 1 x (1/6) + 3 x (1/6) + 4 x (1/6) + 5 x (1/6) + 6 x (1/6)

= 21/6 = 7/2 (1)

$$E(X) = \sum_{i=1}^{6} x_i\, p(x_p)$$

= $x_1^2p(x_1)+x_2^2p(x_2)+x_3^2p(x_3)+x_4^2p(x_4)+x_5^2p(x_5)+x_6p(x_6)$

= 1(1/6) + 4(1/6) + 9(1/6) + 16(1/6) + 25(1/6) + 36(1/6)

$$= \frac{1+4+9+16+25+36}{6} = \frac{91}{6} \qquad (2)$$

Variance (X) = Var (X) = $E(X^2) - [E(X)]^2$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Example :2

Find the value of (i) C (ii) mean of the following distribution given

$$f(x)=\begin{cases} C(x-x^2), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

Given $f(x)=\begin{cases} C(x-x^2), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ (1)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\int_0^1 C(x-x^2)\,dx = 1$ [using (1)] [$\therefore$ 0<x<1]

$$C\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = 1$$

$$C\left[\frac{1}{2}-\frac{1}{3}\right] = 1$$

$$C\left[\frac{3-2}{6}\right] = 1$$

$\frac{C}{6} = 1$ C = 6 (2)

Sub (2) in (1), $f(x) = 6(x - x^2)$, $0 < x < 1$ (3)

Mean $= E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_0^1 x\, 6(x-x^2)\,dx$ [from (3)] [$\therefore$ $0 < x < 1$]

$= \int_0^1 (6x^2 - x^3)\,dx$

$= \left[\frac{6x^3}{3}-\frac{6x^4}{4}\right]_0^1$

$\therefore$ Mean = ½

Mean	C
½	6

1.3 CONTINUOUS DISTRIBUTION FUNCTION

Def :

If f(x) is a p.d.f. of a continuous random variable 'X', then the function

$F_X(x) = F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)\,dx, \ -\infty < x < \infty$

is called the distribution function or cumulative distribution function of the random variable.

1.3.1 PROPERTIES OF CDF OF A R.V. 'X'

(i) $0 \le F(x) \le 1, \ -\infty < x < \infty$

(ii) $\underset{x \to -\infty}{\text{Lt}} F(x) = 0, \qquad \underset{x \to -\infty}{\text{Lt}} F(x) = 1$

(iii) $P(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$

(iv) $F'(x) = \dfrac{dF(x)}{dx} = f(x) \ge 0$

(v) $P(X = x_i) = F(x_i) - F(x_i - 1)$

Example :1.3.1

Given the p.d.f. of a continuous random variable 'X' follows

$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$, find c.d.f. for 'X'

Solution

Given $f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

The c.d.f is $F(x) = \int_{-\infty}^{x} f(x)\,dx, \ -\infty < x < \infty$

(i) When x < 0, then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} 0\,dx = 0$$

(ii) When 0< x < 1, then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_0^x 6x(1-x)\,dx = 6\int_0^x x(1-x)\,dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^x$$

$$= 3x^2 - 2x^3$$

(iii) When x > 1, then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^1 6x(1-x)\,dx + \int_0^x 0\,dx$$

$$= 6\int_0^1 (x - x^2)\,dx = 1$$

Using (1), (2) & (3) we get

$$F(x)=\begin{cases} 0, & x<0 \\ 3x^2-2x^3, & 0<x<1 \\ 1, & x>1 \end{cases}$$

Example :1.3.2

(i) If $f(x)=\begin{cases} e^{-x}, & x\ge 0 \\ 0, & x<0 \end{cases}$ defined as follows a density function ?

(ii) If so determine the probability that the variate having this density will fall in the interval (1, 2).

Solution

Given $f(x)=\begin{cases} e^{-x}, & x\ge 0 \\ 0, & x<0 \end{cases}$

(a) In $(0, \infty)$, e^{-x} is +ve

$\therefore f(x) \ge 0$ in $(0, \infty)$

(b) $\int_{-\infty}^{\infty} f(x)\,dx \quad = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx$

$= \left[-e^{-x}\right]_0^{\infty} \qquad = -e^{-\infty} + 1$

$= 1$

Hence f(x) is a p.d.f

(ii) We know that

$P(a \le X \le b) \qquad = \int_a^b f(x)\,dx$

$P(1 \le X \le 2) \qquad = \int_1^2 f(x)\,dx \qquad = \int_1^2 e^{-x}\,dx = [-e^{-x}]_{+1}^{2}$

$= \int_1^2 e^{-x}\,dx = [-e^{-x}]_{+1}^{2}$

$= -e^{-2} + e^{-1} \qquad = -0.135 + 0.368 \qquad = 0.233$

Example :1.3.3

A probability curve y = f(x) has a range from 0 to ∞. If $f(x) = e^{-x}$, find the mean and variance and the third moment about mean.

Solution

Mean $= \int_0^{\infty} x\, f(x)\,dx$

$= \int_0^{\infty} x e^{-x}\, dx \qquad = \left[x[-e^{-x}] - [e^{-x}] \right]_0^{\infty}$

Mean = 1

Variance $\mu_2 = \int_0^{\infty} (x - \text{Mean})^2 f(x)\, dx$

$= \int_0^{\infty} (x-1)^2 e^{-x}\, dx$

$\mu_2 = 1$

Third moment about mean

$\mu_3 = \int_a^b (x - \text{Mean})^3 f(x)\, dx$

Here a = 0, b = ∞

$\mu_3 = \int_a^b (x-1)^3 e^{-x}\, dx$

$= \left\{ (x-1)^3(-e^{-x}) - 3(x-1)^2(e^{-x}) + 6(x-1)(-e^{-x}) - 6(e^{-x}) \right\}_0^{\infty}$

= -1 + 3 -6 + 6 = 2

$\mu_3 = 2$

1.4 MOMENT GENERATING FUNCTION

Def : The moment generating function (MGF) of a random variable 'X' (about origin) whose probability function f(x) is given by

$M_X(t) \qquad = E[e^{tX}]$

$$= \begin{cases} \int_{x=-\infty}^{\infty} e^{tx} f(x)dx, \text{ for a continuous probably function} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x), \text{ for a discrete probably function} \end{cases}$$

Where t is real parameter and the integration or summation being extended to the entire range of x.

Example: 1.4.1

Prove that the r^{th} moment of the R.V. 'X' about origin is $M_X(t) = \int_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$

Proof

WKT $M_X(t) \quad = E(e^{tX})$

$$= E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + ... + + \frac{(tX)^r}{r!} + ... \right]$$

$$= E[1] + t\,E(X) + \frac{t^2}{2!} E(X^2) + ... + \frac{t^r}{r!} E(X^r) + ...$$

$$M_X(t) \qquad = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + ... + \frac{t^r}{r!}\mu_r' + ...$$

[using $\mu_r' = E(X^r)$]

Thus r^{th} moment = coefficient of $\frac{t^r}{r!}$

Note

1. The above results gives MGF interms of moments.
2. Since $M_X(t)$ generates moments, it is known as moment generating function.

Example :1.4.2

Find μ_1' and μ_2' from $M_X(t)$

Proof

WKT $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

$$M_X(t) = \mu_0' + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots\ldots + \frac{t^r}{r!}\mu_r' \qquad \text{(A)}$$

Differenting (A) W.R.T 't', we get

$$M_X'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \ldots\ldots \qquad \text{(B)}$$

Put t = 0 in (B), we get

$$M_X'(0) = \mu_1' = \text{Mean}$$

Mean $= M_1'(0)$ (or) $\left[\frac{d}{dt}(M_X(t))\right]_{t=0}$

$$M_X''(t) = \mu_2' + t\,\mu_3' + \ldots\ldots.$$

Put t = 0 in (B)

$M_X''(0) = \mu_2'$ (or) $\left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0}$

In general $\mu_r' = \left[\frac{d^r}{dt^r}(M_X(t))\right]_{t=0}$

Example :1.4.3

Obtain the MGF of X about the point X = a.

Proof

The moment generating function of X about the point X = a is

$$M_X(t) = E[e^{t(X-a)}]$$

$$= E\left[1 + t(X-a) + \frac{t^2}{2!}(X-a)^2 + \ldots. + \frac{t^r}{r!}(X-a)^r + \ldots.\right]$$

$$\left[\begin{array}{l}\text{Formula} \\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\end{array}\right]$$

$$= E(1) + E[t(X-a)] + E[\frac{t^2}{2!}(X-a)^2] + \ldots. + E[\frac{t^r}{r!}(X-a)^r] + \ldots.$$

$$=1+tE(X-a)+\frac{t^2}{2!}E(X-a)^2+....+\frac{t^r}{r!}E(X-a)^r+....$$

$$=1+t\mu_1'+\frac{t^2}{2!}\mu_2'+....+\frac{t^r}{r!}\mu_r'+.... \qquad \text{Where } \mu_r'=E[(X-a)^r]$$

$$\left[M_X(t)\right]_{x=a} \quad =1+t\mu_1'+\frac{t^2}{2!}\mu_2'+.....+\frac{t^r}{r!}\mu_r'+.....$$

Result

$$M_{CX}(t)=E[e^{tcx}] \qquad (1)$$

$$M_X(t)=E[e^{ctx}] \qquad (2)$$

From (1) & (2) we get

$$M_{CX}(t)=M_X(ct)$$

Example :1.4.4

If X_1, X_2, ….., X_n are independent variables, then prove that

$$M_{X_1+X_2+....+X_n}(t) \quad =E[e^{t(X_1+X_2+....+X_n)}]$$

$$=E[e^{tX_1}.e^{tX_2}.....e^{tX_n}]$$

$$=E(e^{tX_1}).E(e^{tX_2}).....E(e^{tX_n})$$

[$\therefore$ X_1, X_2, ….., X_n are independent]

$$=M_{X_1}(t).M_{X_2}(t)..........M_{X_n}(t)$$

Example :1.4.5

Prove that if $\cup=\dfrac{X-a}{h}$, then $M_{\cup}(t)=e^{\frac{-at}{h}}.M_X^{\left(\frac{t}{h}\right)}$, where a, h are constants.

Proof

By definition

$$M_{\cup}(t)=E\left[e^{tu}\right] \qquad \because\left[M_X(t)\ =\ E[e^{tx}]\right]$$

$$=E\left[e^{t\left(\frac{X-a}{h}\right)}\right]$$

$$=E\left[e^{\frac{tX}{n}-\frac{ta}{n}}\right]$$

$$=E[e^{\frac{tX}{h}}]\quad E[e^{\frac{-ta}{h}}]$$

$$=e^{\frac{-ta}{h}}E[e^{\frac{tX}{h}}] \qquad \text{[by def]}$$

$$=e^{\frac{-ta}{h}}.\ M_X^{\left(\frac{t}{h}\right)}$$

$\therefore M_{\cup}(t)=e^{\frac{-at}{h}}.M_X\left(\dfrac{t}{h}\right)$, where $\cup=\dfrac{X-a}{h}$ and $M_X(t)$ is the MGF about origin.

Problems

Example: 1.4.6

Find the MGF for the distribution where

$$f(x) = \begin{cases} \frac{2}{3} & \text{at } x = 1 \\ \frac{1}{3} & \text{at } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution

Given $f(1) = \frac{2}{3}$

$f(2) = \frac{1}{3}$

f(3) = f(4) = …… = 0

MGF of a R.V. 'X' is given by

$M_X(t) = E[e^{tx}]$

$= \sum_{x=0}^{\infty} e^{tx} f(x)$

$= e^0 f(0) + e^t f(1) + e^{2t} f(2) + \ldots\ldots$

$= 0 + e^t f(2/3) + e^{2t} f(1/3) + 0$

$= 2/3e^t + 1/3e^{2t}$

$\therefore$ MGF is $M_X(t) = \frac{e^t}{3}[2 + e^t]$

1.5 Discrete Distributions

The important discrete distribution of a random variable 'X' are

1. Binomial Distribution
2. Poisson Distribution
3. Geometric Distribution

1.5.1 BINOMIAL DISTRIBUTION

Def : A random variable X is said to follow binomial distribution if its probability law is given by

$P(x) = p(X = x \text{ successes}) = nC_x p_x q^{n-x}$

Where x = 0, 1, 2, ……., n, p+q = 1

Note

Assumptions in Binomial distribution

i) There are only two possible outcomes for each trail (success or failure).

ii) The probability of a success is the same for each trail.

iii) There are 'n' trails, where 'n' is a constant.

iv) The 'n' trails are independent.

Example: 1.5.1

Find the Moment Generating Function (MGF) of a binomial distribution about origin.

Solution

WKT $M_X(t) = \sum_{x=0}^{n} e^{tx} p(x)$

Let 'X' be a random variable which follows binomial distribution then MGF about origin is given by

$$E[e^{tX}] = M_X(t) = \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x} \qquad \left[\because p(x) = nC_x p^x q^{n-x}\right]$$

$$= \sum_{x=0}^{n} (e^{tx}) p^x nC_x q^{n-x}$$

$$= \sum_{x=0}^{n} (pe^t)^x nC_x q^{n-x}$$

$$\therefore M_X(t) = (q+pe^t)^n$$

Example :1.5. 2

Find the mean and variance of binomial distribution.

Solution

$$M_X(t) = (q+pe^t)^n$$

$$\therefore M_X^{'}(t) = n(q+pe^t)^{n-1}.pe^t$$

Put t = 0, we get

$$M_X^{'}(0) = n(q+p)^{n-1}.p$$

Mean = E(X) = np $\quad [\because (q+p)=1] \quad \left[\text{Mean } M_X^{'}(0)\right]$

$$M_X^{''}(t) = np\left[(q+pe^t)^{n-1}.e^t + e^t(n-1)(q+pe^t)^{n-2}.pe^t\right]$$

Put t = 0, we get

$$M_X^{''}(t) = np\left[(q+p)^{n-1} + (n-1)(q+p)^{n-2}.p\right]$$

$$= np\left[1+(n-1)p\right]$$

$$= np + n^2p^2 - np^2$$

$$= n^2p^2 + np(1-p)$$

$$M_X^{''}(0) = n^2p^2 + npq \qquad [\because 1-p=q]$$

$$M_X^{''}(0) = E(X^2) = n^2p^2 + npq$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = n^2/p^2 + npq - n^2/p^2 = npq$$

Var (X) = npq

S.D = $\sqrt{npq}$

Example :1.5.3

Find the Moment Generating Function (MGF) of a binomial distribution about mean (np).

Solution

Wkt the MGF of a random variable X about any point 'a' is

$M_x(t)$ (about X = a) $= E[e^{t(X-a)}]$
Here 'a' is mean of the binomial distribution
$M_X(t)$ (about X = np) $= E[e^{t(X-np)}]$
$= E[e^{tX} . e^{-tnp)}]$
$= e^{-tnp} . [-[e^{-tX)}]]$
$= e^{-tnp} . (q+pe^t)-^n$
$= (e^{-tp})^n. (q + pe^t)^n$
$\therefore$ MGF about mean $= (e^{-tp})^n. (q + pe^t)^n$

Example :1.5.4
Additive property of binomial distribution.
Solution
The sum of two binomial variants is not a binomial variate.
Let X and Y be two independent binomial variates with parameter (n_1, p_1) and (n_2, p_2) respectively.
Then

$$M_X(t) = \left(q_1 + p_1e^t\right)^{n_1}, \qquad M_Y(t) = \left(q_2 + p_2e^t\right)^{n_2}$$

$$\therefore \ M_{X+Y}(t) = M_X(t).M_Y(t) \qquad [\because X \ \& \ Y \text{ are independent R.V.'s}]$$

$$= \left(q_1 + p_1e^t\right)^{n_1} . \left(q_2 + p_2e^t\right)^{n_2}$$

RHS cannot be expressed in the form $\left(q + pe^t\right)^n$. Hence by uniqueness theorem of MGF X+Y is not a binomial variate. Hence in general, the sum of two binomial variates is not a binomial variate.

Example :1.5.5

If $M_X(t) = \left(q+pe^t\right)^{n_1}$, $M_Y(t) = \left(q+pe^t\right)^{n_2}$, then

$$M_{X+Y}(t) = \left(q+pe^t\right)^{n_1+n_2}$$

Problems on Binomial Distribution
1. Check whether the following data follow a binomial distribution or not. Mean = 3; variance = 4.
Solution
Given Mean np = 3 (1)
Variance npr = 4 (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{np}{npq} = \frac{3}{4}$$

$$\Rightarrow q = \frac{4}{3} = 1\frac{1}{3} \text{ which is} > 1.$$

Since q > 1 which is not possible (0 < q < 1). The given data not follow binomial distribution.

Example :1.5.6

The mean and SD of a binomial distribution are 5 and 2, determine the distribution.

Solution

Given Mean = np = 5 (1)

SD = $\sqrt{npq}$ = 2 (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{np}{npq} = \frac{4}{5} \Rightarrow q = \frac{4}{5}$$

$$\therefore p = 1 - \frac{4}{5} = \frac{1}{5} \Rightarrow p = \frac{1}{5}$$

Sub (3) in (1) we get

n x 1/5 = 5

n = 25

$\therefore$ The binomial distribution is

P(X = x) = p(x) = $nC_x p^x q^{n-x}$

= $25C_x(1/5)^x(4/5)^{n-x}$, x = 0, 1, 2, ….., 25

1.6 Passion Distribution

Def :

A random variable X is said to follow if its probability law is given by

$$P(X = x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots, \infty$$

Poisson distribution is a limiting case of binomial distribution under the following conditions or assumptions.

1. The number of trails 'n' should e infinitely large i.e. $n \to \infty$.
2. The probability of successes 'p' for each trail is infinitely small.
3. np = λ, should be finite where λ is a constant.

*** To find MGF**

$M_X(t)$ = $E(e^{tx})$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \left(\frac{\lambda^x e^{\lambda}}{x!} \right)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left[1 + \lambda e^t + \frac{(\lambda e^t)^2}{2!} + \ldots\ldots \right]$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

Hence $M_X(t) = e^{\lambda(e^t-1)}$

* To find Mean and Variance

WKT $M_X(t) = e^{\lambda(e^t-1)}$

$\therefore \quad M_X'(t) = e^{\lambda(e^t-1)}.e^t$

$M_X'(0) = e^{-\lambda}.\lambda$

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x.p(x)$$

$$= \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{x.\, e^{-\lambda}\lambda\lambda^{x-1}}{x!}$$

$$= 0 + e^{-\lambda}.\lambda\sum_{x=1}^{\infty} \frac{x.\lambda^{x-1}}{x!}$$

$$= \lambda e^{-\lambda}.\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{2!}+.....\right]$$

$$= \lambda e^{-\lambda}.e^{\lambda}$$

Mean = λ

$$\mu_2' = E[X^2] = \sum_{x=0}^{\infty} x^2.p(x) = \sum_{x=0}^{\infty} x^2.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \{x(x-1)+x\}.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{x.e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^2\sum_{x=0}^{\infty} \frac{\lambda^{x-2}}{(x-2)(x-3)....1} + \lambda$$

$$= e^{-\lambda}\lambda^2\sum_{x=0}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+....\right] + \lambda$$

$$= \lambda^2 + \lambda$$

Variance $\mu_2 = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Variance = λ

Hence Mean = Variance $=\lambda$

Note : * sum of independent Poisson Vairates is also Poisson variate.

1.6.1 PROBLEMS ON POISSON DISTRIBUTION

Example:1.6.2

If x is a Poisson variate such that P(X=1) = $\frac{3}{10}$ and P(X=2) = $\frac{1}{5}$, find the P(X=0) and P(X=3).

Solution

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\therefore P(X=1) = e^{-\lambda}\lambda = \frac{3}{10} \qquad \text{(Given)}$$

$$= \lambda e^{-\lambda} = \frac{3}{10} \qquad (1)$$

$$P(X=2) = \frac{e^{-\lambda}\lambda^2}{2!} = \frac{1}{5} \qquad \text{(Given)}$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = \frac{1}{5} \qquad (2)$$

$$(1) \Rightarrow e^{-\lambda}\lambda = \frac{3}{10} \qquad (3)$$

$$(2) \Rightarrow e^{-\lambda}\lambda^2 = \frac{2}{5} \qquad (4)$$

$$\frac{(3)}{(4)} \Rightarrow \frac{1}{\lambda} = \frac{3}{4}$$

$$\lambda = \frac{4}{3}$$

$$\therefore \quad P(X=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-4/3}$$

$$P(X=3) = \frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-4/3}(4/3)^3}{3!}$$

Example :1.6.3

If X is a Poisson variable

P(X = 2) = 9 P(X = 4) + 90 P(X=6)

Find (i) Mean if X (ii) Variance of X

Solution

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,2,.....$$

Given P(X = 2) = 9 P(X = 4) + 90 P(X=6)

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$
$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$
$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$
$$\lambda^4 + 3\lambda^2 - 4 = 0$$
$\lambda^2 = 1$ or $\lambda^2 = -4$

$\lambda = \pm 1$ or $\lambda = \pm 2i$

$\therefore$ Mean = λ = 1, Variance = λ = 1

$\therefore$ Standard Deviation = 1

Example :1.6.4 Derive probability mass function of Poisson distribution as a limiting case of Binomial distribution

Solution

We know that the Binomial distribution is

$P(X=x) = nC_x\, p^x q^{n-x}$

$$= \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$
$$= \frac{1.2.3.......(n-x)(n-x+1)......np^n}{1.2.3.....(n-x)\ \ x!} \frac{(1-p)^n}{(1-p)^x}$$
$$= \frac{1.2.3.......(n-x)(n-x+1)......n}{1.2.3.....(n-x)\ \ x!} \left(\frac{p}{1-p}\right)^x (1-p)^n$$
$$= \frac{n(n-1)(n-2)......(n-x+1)}{x!} \frac{\lambda^x}{n^x} \frac{1}{\left(1-\frac{\lambda}{n}\right)^x} \left(1-\frac{\lambda}{n}\right)^n$$
$$= \frac{n(n-1)(n-2)......(n-x+1)}{x!}\ x \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$
$$P(X=x) = \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)......\left\{1-\left(\frac{x-1}{n}\right)\right\}}{x!} \lambda^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$
$$= \frac{\lambda^x}{x!} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)......\left\{1-\left(\frac{x-1}{n}\right)\right\}\left(1-\frac{\lambda}{n}\right)^{n-x}$$
When $n \to \infty$
$$P(X=x) = \frac{\lambda^x}{x!} \lim_{n\to\infty} \left[1-\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)......\left\{1-\left(\frac{x-1}{n}\right)\right\}\left(1-\frac{\lambda}{n}\right)^{n-x}\right]$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty}\left(1-\frac{1}{n}\right) \lim_{n\to\infty}\left(1-\frac{2}{n}\right) \ldots\ldots \lim_{n\to\infty} 1-\left(\frac{x-1}{n}\right)$$

We know that

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x} = e^{-\lambda}$$

and $\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{2}{n}\right)\ldots\ldots = \lim_{n\to\infty}\left(1-\left(\frac{x-1}{n}\right)\right) = 1$

$\therefore$ P(X=x) $= \frac{\lambda^x}{x!} e^{-\lambda}, x = 0,1,2,\ldots\ldots\infty$

1.7 GEOMETRIC DISTRIBUTION

Def: A discrete random variable 'X' is said to follow geometric distribution, if it assumes only non-negative values and its probability mass function is given by

$P(X=x) = p(x) = q^{x-1}$; x = 1, 2, ……., 0 < p < 1, Where q = 1-p

Example:1.7.1

To find MGF

$M_X(t) = E[e^{tx}]$

$= \sum e^{tx} p(x)$

$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$

$= \sum_{x=1}^{\infty} e^{tx} q^{x} q^{-1} p$

$= \sum_{x=1}^{\infty} e^{tx} q^{x} p / q$

$= p / q \sum_{x=1}^{\infty} e^{tx} q^{x}$

$= p / q \sum_{x=1}^{\infty} (e^{t} q)^{x}$

$= p / q \left[(e^t q)^1 + (e^t q)^2 + (e^t q)^3 + \ldots\right]$

Let $x = e^t q$ $= p / q \left[x + x^2 + x^3 + \ldots\right]$

$= \frac{p}{q} x \left[1 + x + x^2 + \ldots\right]$ $= \frac{p}{q}(1-x)^{-1}$

$= \frac{p}{q} qe^t \left[1 - qe^t\right]$ $= pe^t [1-qe^t]^{-1}$

$\therefore M_X(t) = \frac{pe^t}{1-qe^t}$

*** To find the Mean & Variance**

$M_X^{'}(t) = \dfrac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2} = \dfrac{pe^t}{(1-qe^t)^2}$

$\therefore$ E(X) = $M_X^{'}(0)$ = 1/p

$\therefore$ Mean = 1/p

Variance $\mu_X^{''}(t) = \dfrac{d}{dt}\left[\dfrac{pe^t}{(1-qe^t)^2}\right]$

$= \dfrac{(1-qe^t)^2 pe^t - pe^t 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$

$= \dfrac{(1-qe^t)^2 pe^t + 2pe^t qe^t (1-qe^t)}{(1-qe^t)^4}$

$M_X^{''}(0) = \dfrac{1+q}{p^2}$

Var (X) = E(X^2) – [E(X)]^2 = $\dfrac{(1+q)}{p^2} - \dfrac{1}{p^2} \Rightarrow \dfrac{q}{p^2}$

Var (X) = $\dfrac{q}{p^2}$

Note

Another form of geometric distribution

P[X=x] = $q^x p$; x = 0, 1, 2, ….

$M_X(t) = \dfrac{p}{(1-qe^t)}$

Mean = q/p, Variance = q/p^2

Example 1.7.2

If the MGF of X is $(5\text{-}4et)^{-1}$, find the distribution of X and P(X=5)

Solution

Let the geometric distribution be

P(X = x) = $q^x p$, x = 0, 1, 2, …..

The MGF of geometric distribution is given by

$\dfrac{p}{1-qe^t}$ (1)

Here $M_X(t) = (5 - 4e^t)^{-1} \Rightarrow 5^{-1}\left[1-\dfrac{4}{5}e^t\right]^1$ (2)

Company (1) & (2) we get $q = \dfrac{4}{5}$; $p = \dfrac{1}{5}$

$\therefore$ P(X = x) = pq^x, x = 0, 1, 2, 3, …….

$= \left(\dfrac{1}{5}\right)\left(\dfrac{4}{5}\right)^x$

$P(X = 5) \quad = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^5 = \frac{4^5}{5^6}$

1.8 CONTINUOUS DISTRIBUTIONS

If ‘X’ is a continuous random variable then we have the following distribution

1. Uniform (Rectangular Distribution)
2. Exponential Distribution
3. Gamma Distribution
4. Normal Distribution

1.8.1. Uniform Distribution (Rectangular Distribution)

Def : A random variable X is set to follow uniform distribution if its

$$f(x)=\begin{cases}\frac{1}{b-a}, & a<x<b\\ 0, & \text{otherwise}\end{cases}$$

*** To find MGF**

$$M_X(t) \quad = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_a^b e^{tx}\frac{1}{b-a}dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_b^a$$

$$= \frac{1}{(b-a)t}\left[e^{bx}-e^{at}\right]$$

$\therefore$ The MGF of uniform distribution is

$$M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$$

* To find Mean and Variance

$$E(X) \quad = \int_{-\infty}^{\infty} x\ f(x)dx$$

$$= \int_a^b b_x \frac{1}{b-a}dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{\left(\frac{x^2}{2}\right)_a^b}{b-a}$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2} = \frac{a+b}{2}$$

Mean $\mu_1' = \frac{a+b}{2}$

Putting r = 2 in (A), we get

$$\mu_2' \quad = \int_a^b x^2\ f(x)dx = \int_a^b \frac{x^2}{b-a}dx$$

$$= \frac{a^2 + ab + b^2}{3}$$

$\therefore$ Variance $= \mu_2' - \mu_1'^2$

$$= \frac{b^2 + ab + b^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12}$$

$$\text{Variance} = \frac{(b-a)^2}{12}$$

PROBLEMS ON UNIFORM DISTRIBUTION

Example 1

If X is uniformly distributed over $(-\alpha, \alpha)$, $\alpha < 0$, find α so that

(i) $P(X>1) = 1/3$

(ii) $P(|X| < 1) = P(|X| > 1)$

Solution

If X is uniformly distributed in $(-\alpha, \alpha)$, then its p.d.f. is

$$f(x) = \begin{cases} \dfrac{1}{2\alpha} & -\alpha < x < \alpha \\ 0 & \text{otherwise} \end{cases}$$

(i) $P(X>1) = 1/3$

$$\int_1^{\alpha} f(x)dx = 1/3$$

$$\int_1^{\alpha} \frac{1}{2\alpha} dx = 1/3$$

$$\frac{1}{2\alpha}(x)_1^{\alpha} = 1/3 \qquad \Rightarrow \frac{1}{2\alpha}(\alpha - 1) = 1/3$$

$\alpha = 3$

(ii) $P(|X| < 1) = P(|X| > 1) = 1 - P(|X| < 1)$

$P(|X| < 1) + P(|X| < 1) = 1$

$2\, P(|X| < 1) = 1$

$2\, P(-1 < X < 1) = 1$

$$2\int_1^1 f(x)dx = 1$$

$$\cancel{2}\int_1^1 \frac{1}{\cancel{2}\alpha} dx = 1$$

$\Rightarrow \alpha = 2$

Note

1. The distribution function F(x) is given by

$$F(x) \quad = \begin{cases} 0 & -\alpha < x < \alpha \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & b < x < \infty \end{cases}$$

2. The p.d.f. of a uniform variate 'X' in (-a, a) is given by

$$F(x) \quad = \begin{cases} \dfrac{1}{2a} & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

1.8.2 THE EXPONENTIAL DISTRIBUTION

Def : A continuous random variable 'X' is said to follow an exponential distribution with parameter λ>0 if its probability density function is given by

$$F(x) \quad = \begin{cases} \lambda e^{-\lambda x} & x > a \\ 0 & \text{otherwise} \end{cases}$$

To find MGF

Solution

$$M_X(t) \qquad = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx \qquad = \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{\lambda - t} \right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda - t)} \left[e^{-\infty} - e^{-0} \right] \qquad = \frac{\lambda}{\lambda - t}$$

$\therefore$ MGF of x = $\dfrac{\lambda}{\lambda - t}$, $\lambda > t$

*** To find Mean and Variance**

We know that MGF is

$$M_X(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1 - \frac{t}{\lambda}} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \ldots.. + \frac{t^r}{\lambda^r}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{2!}\left(\frac{2!}{\lambda^2}\right) + \ldots.. + \frac{t^r}{r!}\left(\frac{t!}{\lambda^r}\right)$$

$M_X(t) = \sum_{r=0}^{\infty}\left(\frac{t}{\lambda}\right)^r$

$\therefore$ Mean $\quad \mu_1' = \text{Coefficient of } \frac{t^1}{1!} = \frac{1}{\lambda}$

$\mu_2' = \text{Coefficient of } \frac{t^2}{2!} = \frac{2}{\lambda^2}$

Variance= $\mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Variance = $\frac{1}{\lambda^2}$ $\qquad$ Mean = $\frac{1}{\lambda}$

Example 1

Let 'X' be a random variable with p.d.f

$$F(x) \quad = \begin{cases} \frac{1}{3}e^{\frac{-x}{3}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\qquad$ 1) P(X > 3) $\qquad$ 2) MGF of 'X'

Solution

WKT the exponential distribution is

F(x) = $\lambda e^{-\lambda x}$, $\quad x > 0$

Here $\lambda = \frac{1}{3}$

P(x>3) = $\int_3^{\infty} f(x)\,dx \qquad = \int_3^{\infty} \frac{1}{3} e^{-\frac{x}{3}}\,dx$

P(X>3) = e^{-1}

MGF is $\qquad M_X(t) \qquad = \frac{\lambda}{\lambda - t}$

$$= \frac{\frac{1}{3}}{\frac{1}{3} - t} \qquad = \frac{\frac{1}{3}}{\frac{1-3t}{3}} \qquad = \frac{1}{1-3t}$$

$M_X(t) = \frac{1}{1-3t}$

Note

If X is exponentially distributed, then

P(X > s+t / x > s) $\qquad$ = P(X > t), for any s, t > 0.

1.8.3 GAMMA DISTRIBUTION

Definition

A Continuous random variable X taking non-negative values is said to follow gamma distribution , if its probability density function is given by

$f(x) =$ ——— , $\alpha>0, 0 < x < \infty$

=0, elsewhere

and = dx

=0, elsewhere

When α is the parameter of the distribution.

Additive property of Gamma Variates

If $X_1, X_2, X_{3,\ldots} X_k$ are independent gamma variates with parameters $\lambda_1, \lambda_2, \ldots \lambda_k$ respectively then $X1+X2+X_{3+\ldots}+X_k$ is also a gamma variates with parameter $\lambda_1+\lambda_2+\ldots+\lambda_k$.

Example :1

Customer demand for milk in a certain locality ,per month , is Known to be a general Gamma R V .If the average demand is a liters and the most likely demand b liters (b<a) , what is the varience of the demand?

Solution :

Let X be represent the monthly Customer demand for milk.

Average demand is the value of E(X).

Most likely demand is the value of the mode of X or the value of X for which its density function is maximum.

If f(x) is the its density function of X ,then

$f(x) =$ —— . $x^{k-1} e^{-\lambda x}$ $e^{-\lambda x}$, $x>0$

$f(x) =$ —— .[$(k-1) x^{k-2} e^{-\lambda x}$ - $e^{-\lambda x}$]

= 0 ,when x=0 , x=——

$f''(x) =$ —— .[$(k-1) x^{k-2} e^{-\lambda x}$ - $e^{-\lambda x}$]

<0 , when x=——

Therefour f(x) is maximum , when x=——

i.e ,Most likely demand =——= b ….(1)

and E(X) = – ………(2)

Now V(X) = = —= – –

= a (a-b) From (1) and (2)

TUTORIAL QUESTIONS

1.It is known that the probability of an item produced by a certainmachine will be defective is 0.05. If the produced items are sent to themarket in packets of 20, fine the no. of packets containing at least,exactly and atmost 2 defective items in a consignment of 1000 packetsusing (i) Binomial distribution (ii) Poisson approximation to binomialdistribution.

2. The daily consumption of milk in excess of 20,000 gallons isapproximately exponentially distributed with . 3000 = θ The city has adaily stock of 35,000 gallons. What is the probability that of two daysselected at random, the stock is insufficient for both days.

3.The density function of a random variable X is given by f(x)= KX(2-X), $0 \le X \le 2$.Find K, mean, variance and r^{th} moment.

4.A binomial variable X satisfies the relation 9P(X=4)=P(X=2) when n=6. Find the parameter p of the Binomial distribution.

5. Find the M.G.F for Poisson Distribution.

6. If X and Y are independent Poisson variates such that P(X=1)=P(X=2) and P(Y=2)=P(Y=3). Find V(X-2Y).

7.A discrete random variable has the following probability distribution

X:	0	1	2	3	4	5	6	7	8
P(X)	a	3a	5a	7a	9a	11a	13a	15a	17a

Find the value of a, P(X<3) and c.d.f of X.

7. In a component manufacturing industry, there is a small probability of 1/500 for any component to be defective. The components are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (1). No defective. (2). Two defective components in a consignment of 10,000 packets.

WORKED OUT EXAMPLES

Example 1

Given the p.d.f. of a continuous random variable ‘X’ follows

$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$, find c.d.f. for ‘X’

Solution

Given $f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

The c.d.f is $F(x) = \int_{-\infty}^{x} f(x)\,dx, \ -\infty < x < \infty$

(i) When x < 0, then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} 0 \, dx \qquad = 0$$

(ii) When 0< x < 1, then

$$F(x) \quad = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} 6x(1-x)\,dx \quad = 6\int_{0}^{x} x(1-x)\,dx \ = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^x$$

$$= 3x^2 - 2x^3$$

(iii) When x > 1, then

$$F(x) \quad = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 6x(1-x)\,dx + \int_{0}^{x} 0 \, dx$$

$$= 6 \int_{0}^{1} (x - x^2)\,dx \qquad = 1$$

Using (1), (2) & (3) we get

$$F(x) = \begin{cases} 0, & x < 0 \\ 3x^2 - 2x^3, & 0 < x < 1 \\ 1, & x > 1 \end{cases}$$

Example :2

A random variable X has the following probability function

Values of X									
Probability P(X)		a	a	a	a	1a	3a	5a	7a

(i) Determine the value of ‘a’
(ii) Find P(X<3), P(X≥3), P(0<X<5)
(iii) Find the distribution function of X.

Solution

Table 1

Values of X									
p(x)		a	a	a	a	1a	3a	5a	7a

(i) We know that if p(x) is the probability of mass function then

$$\sum_{i=0}^{8} p(x_i) = 1$$

p(0) + p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + p(8) = 1
a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1
81 a = 1
a = 1/81
put a = 1/81 in table 1, e get table 2

Table 2

X = x									
P (x)	/81	/81	/81	/81	/81	1/81	3/81	5/81	7/81

(ii) $P(X < 3)$ = p(0) + p(1) + p(2)
= 1/81+ 3/81 + 5/81 = 9/81
(ii) $P(X \geq 3)$ = 1 - p(X < 3)
= 1 - 9/81 = 72/81
(iii) $P(0 < x < 5)$ = p(1) + p(2) + p(3) + p(4) here 0 & 5 are not include
= 3/81 + 5/81 + 7/81 + 9/81

$$= \frac{3 + 5 + 7 + 8 + 9}{81} = \frac{24}{81}$$

(iv) To find the distribution function of X using table 2, we get

X = x	**F(X) = P(x ≤ x)**
0	F(0) = p(0) = 1/81
1	F(1) = P(X ≤ 1) = p(0) + p(1) = 1/81 + 3/81 = 4/81
2	F(2) = P(X ≤ 2) = p(0) + p(1) + p(2) = 4/81 + 5/81 = 9/81
3	F(3) = P(X ≤ 3) = p(0) + p(1) + p(2) + p(3) = 9/81 + 7/81 = 16/81
4	F(4) = P(X ≤ 4) = p(0) + p(1) + …. + p(4) = 16/81 + 9/81 = 25/81
5	F(5) = P(X ≤ 5) = p(0) + p(1) + ….. + p(4) + p(5) = 2/81 + 11/81 = 36/81
6	F(6) = P(X ≤ 6) = p(0) + p(1) + ….. + p(6) = 36/81 + 13/81 = 49/81

7	$F(7) = P(X \le 7) = p(0) + p(1) + \dots + p(6) + p(7)$ $= 49/81 + 15/81 = 64/81$
8	$F(8) = P(X \le 8) = p(0) + p(1) + \dots + p(6) + p(7) + p(8)$ $= 64/81 + 17/81 = 81/81 = 1$

Example :3

The mean and SD of a binomial distribution are 5 and 2, determine the distribution.

Solution

Given Mean = np = 5 (1)

SD = $\sqrt{npq} = 2$ (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{np}{npq} = \frac{4}{5} \Rightarrow q = \frac{4}{5}$$

$$\therefore p = 1 - \frac{4}{5} = \frac{1}{5} \Rightarrow p = \frac{1}{5}$$

Sub (3) in (1) we get

n x 1/5 = 5

n = 25

$\therefore$ The binomial distribution is

$P(X = x) = p(x) = nC_x p^x q^{n-x}$

$= 25C_x (1/5)^x (4/5)^{n-x}$, $x = 0, 1, 2, \dots, 25$

Example :4

If X is a Poisson variable

$P(X = 2) = 9 P(X = 4) + 90 P(X=6)$

Find (i) Mean if X (ii) Variance of X

Solution

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \ x = 0,1,2,\dots..$$

Given $P(X = 2) = 9 P(X = 4) + 90 P(X=6)$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$\lambda^2 = 1$ or $\lambda^2 = -4$

$\lambda = \pm 1$ or $\lambda = \pm 2i$

$\therefore$ Mean = λ = 1, Variance = λ = 1

$\therefore$ Standard Deviation = 1

UNIT – II

TWO DIMENSIONAL RANDOM VARIABLES

Introduction

In the previous chapter we studied various aspects of the theory of a single R.V. In this chapter we extend our theory to include two R.V's one for each coordinator axis X and Y of the XY Plane.

DEFINITION : Let S be the sample space. Let X = X(S) & Y = Y(S) be two functions each assigning a real number to each outcome $s \in S$. hen (X, Y) is a two dimensional random variable.

Types of random variables

1. Discrete R.V.'s
2. Continuous R.V.'s

2.1 Discrete R.V.'s (Two Dimensional Discrete R.V.'s)

If the possible values of (X, Y) are finite, then (X, Y) is called a two dimensional discrete R.V. and it can be represented by (x_i, y), i = 1,2,….,m.

In the study of two dimensional discrete R.V.'s we have the following 5 important terms.

- Joint Probability Function (JPF) (or) Joint Probability Mass Function.
- Joint Probability Distribution.
- Marginal Probability Function of X.
- Marginal Probability Function of Y.
- Conditional Probability Function.

2.1.1 Joint Probability Function of discrete R.V.'s X and Y

The function $P(X = x_i, Y = y_j) = P(x_i, y_j)$ is called the joint probability function for discrete random variable X and Y is denote by p_{ij}.

Note

1. $P(X = x_i, Y = y_j) = P[(X = x_i)\cap(Y = y_j)] = p_{ij}$
2. It should satisfies the following conditions

(i) $p_{ij} \geq \forall$ i, j (ii) $\Sigma_j\Sigma_i\, p_{ij} = 1$

2.1.2 Marginal Probability Function of X

If the joint probability distribution of two random variables X and Y is given then the marginal probability function of X is given by

$P_x(x_i) = p_i$ (marginal probability function of Y)

2.1.3 Conditional Probabilities

The conditional probabilities function of X given $Y = y_j$ is given by

$$P[X = x_i / Y = y_j] = \frac{P[X = x_i / Y = y_j]}{P[Y = y_j]} = \frac{p_{ij}}{p_{.j}}$$

The set $\{x_i, p_{ij} / p_{.j}\}$, i = 1, 2, 3, …..is called the conditional probability distribution of X given $Y = y_j$.

The conditional probability function of Y given $X = x_i$ is given by

$$P[Y = y_i / X = x_j] = \frac{P[Y = y_i / X = x_j]}{P[X = x_j]} = \frac{p_{ij}}{p_{i.}}$$

The set $\{y_i, p_{ij} / p_{i.}\}$, j = 1, 2, 3, …..is called the conditional probability distribution of Y given $X = x_i$.

SOLVED PROBLEMS ON MARGINAL DISTRIBUTION

Example:2.1(a)

From the following joint distribution of X and Y find the marginal distributions.

Y \ X	0	1	2
0	3/28	9/28	3/28
1	3/14	3/14	0
2	1/28	0	0

Solution

Y \ X	0	2	$P_Y(y)$ = p(Y=y)
0	3/28	3/2	15/28 =

	P(0,0)	8 P(2,0)	$P_y(0)$
1	3/14 P(0, 1)	3/1 4 P(1,1)	6/14 = $P_y(1)$
2	1/28 P(0,2)	0 P(2,2)	1/28 = $P_y(2)$
$P_X(X)$ = P(X=x)	10/28 = 5/14 $P_X(0)$	3/2 8 $P_X($ 2)	1

The marginal distribution of X

$P_X(0)$ = P(X = 0) = p(0,0) + p(0,1) + p(0,2) = 5/14

$P_X(1)$ = P(X = 1) = p(1,0) + p(1,1) + p(1,2) = 15/28

$P_X(2)$ = P(X = 2) = p(2,0) + p(2,1) + p(2,2) = 3/28

Marginal probability function of X is

$$P_X(x) = \begin{cases} \frac{5}{14}, & x = 0 \\ \frac{15}{28}, & x = 1 \\ \frac{3}{28}, & x = 2 \end{cases}$$

The marginal distribution of Y

$P_Y(0)$ = P(Y = 0) = p(0,0) + p(1,0) + p(2,0) = 15/28

$P_Y(1)$ = P(Y = 1) = p(0,1) + p(2,1) + p(1,1) = 3/7

$P_Y(2)$ = P(Y = 2) = p(0,2) + p(1,2) + p(2,2) = 1/28

Marginal probability function of Y is

$$P_Y(y) = \begin{cases} \frac{15}{28}, & y = 0 \\ \frac{3}{7}, & y = 1 \\ \frac{1}{28}, & y = 2 \end{cases}$$

2.2 CONTINUOUS RANDOM VARIABLES

- Two dimensional continuous R.V.'s

If (X, Y) can take all the values in a region R in the XY plans then (X, Y) is called two-dimensional continuous random variable.

- Joint probability density function :

(i) $f_{XY}(x,y) \geq 0$; (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dydx = 1$

- Joint probability distribution function

$F(x,y) = P[X \leq x, Y \leq y]$

$$= \int_{-\infty}^{x}\left\{\int_{-\infty}^{y} f(x,y)dx\right\}dy$$

- Marginal probability density function

$f(x) = f_X(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)dy$ (Marginal pdf of X)

$f(y) = f_Y(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)dy$ (Marginal pdf of Y)

- Conditional probability density function

(i) $P(Y = y / X = x) = f(y/x) = \dfrac{f(x,y)}{f(x)}, \; f(x) > 0$

(ii) $P(X = x / Y = y) = f(x/y) = \dfrac{f(x,y)}{f(y)}, \; f(y) > 0$

Example :2.2.1

Show that the function $f(x,y)=\begin{cases}\frac{2}{5}(2x+3y), & 0<x<1, \quad 0<y<1\\ 0 & \text{otherwise}\end{cases}$

is a joint density function of X and Y.

Solution

We know that if f(x,y) satisfies the conditions

(i) $f(x,y)\geq 0$ (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)=1$, then f(x,y) is a jdf

Given $f(x,y)=\begin{cases}\frac{2}{5}(2x+3y), & 0<x<1, \quad 0<y<1\\ 0 & \text{otherwise}\end{cases}$

(i) $f(x,y)\geq 0$ in the given interval $0 \leq (x,y) \leq 1$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$

$$= \frac{2}{5}\int_0^1\int_0^1\left[2\frac{x^2}{2}+3xy\right]_0^1 dy$$

$$= \frac{2}{5}\int_0^1(1+3y)\,dy = \frac{2}{5}\left[y+\frac{3y^2}{2}\right]_0^1 = \frac{2}{5}\left(1+\frac{3}{2}\right)$$

$$= \frac{2}{5}\left(\frac{5}{2}\right)=1$$

Since f(x,y) satisfies the two conditions it is a j.d.f.

Example: 2.2.2

The j.d.f of the random variables X and Y is given

$f(x,y)=\begin{cases}8xy, & 0<x<1, \quad 0<y<x\\ 0, & \text{otherwise}\end{cases}$

Find (i) $f_X(x)$ (ii) $f_Y(y)$ (iii) f(y/x)

Solution

We know that

(i) The marginal pdf of 'X' is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 8xy\,dy = 4x^3$$

$$f(x)=4x^3,\ 0<x<1$$

(ii) The marginal pdf of 'Y' is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 8xy\,dy = 4y$$

$$f(y) = 4y,\ 0 < y < \alpha$$

(iii) We know that

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{8xy}{4x^3} = \frac{2y}{x^2},\ 0 < y < x,\ 0 < x < 1$$

Result

Marginal pdf g	Marginal pdf y	F(y/x)
$4x^3$, $0<x<1$	$4y$, $0<y<x$	$\frac{2y}{x^2}, 0 < y < x, 0 <$

2.3 REGRESSION

* Line of regression

The line of regression of X on Y is given by

$$x - \bar{x} = r.\frac{\sigma y}{\sigma x}(y - \bar{y})$$

The line of regression of Y on X is given by

$$y - \bar{y} = r.\frac{\sigma y}{\sigma x}(x - \bar{x})$$

* Angle between two lines of Regression.

$$\tan\theta = \frac{1 - r^2}{r}\left(\frac{\sigma_y \sigma_x}{\sigma_{x^2} + \sigma_{y^2}}\right)$$

* Regression coefficient

Regression coefficients of Y on X

$$r.\frac{\sigma y}{\sigma x} = b_{YX}$$

Regression coefficient of X and Y

$$r.\frac{\sigma x}{\sigma y} = b_{XY}$$

$\therefore$ Correlation coefficient $r = \pm\sqrt{b_{XY} \times b_{YX}}$

Problems

1. From the following data, find

(i) The two regression equation

(ii) The coefficient of correlation between the marks in Economic and Statistics.

(iii) The most likely marks in statistics when marks in Economic are 30.

Marks in Economics	5	8	5	2	1	6	9	8	4	2
Marks in Statistics	0	6	9	1	6	2	1	0	3	9

Solution

		$X-\overline{X}=x$	$X-\overline{Y}=y$	$(X$	$(Y$	$(X-\overline{X})^2$ $(Y-\overline{Y})$
5	3	-7	5	49	25	-35
8	6	-4	8	16	64	-32
5		3	11	9	12 1	33
2	1	0	3	0	9	0
1	6	-1	-2	1	4	2
6	2	4	-6	16	36	-24
9	1	-3	-7	09	49	+21
8	0	6	-8	36	64	-48
4	3	2	-5	4	25	-48
2	9	0	1	0	1	100
20	80	0	0	14 0	39 8	-93

Here $\overline{X}=\frac{\sum X}{n}=\frac{320}{10}=32$ and $\overline{Y}=\frac{\sum Y}{n}=\frac{380}{10}=38$

Coefficient of regression of Y on X is

$$b_{YX}=\frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sum(X-\overline{X})^2}=\frac{-93}{140}=-0.6643$$

Coefficient of regression of X on Y is

$$b_{XY}=\frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sum(Y-\overline{Y})^2}=\frac{-93}{398}=-0.2337$$

Equation of the line of regression of X and Y is

$X - \overline{X} \qquad = b_{XY}(Y - \overline{Y})$

$X - 32 \qquad = -0.2337\ (y - 38)$

$X \qquad = -0.2337\ y + 0.2337 \times 38 + 32$

$X \qquad = -0.2337\ y + 40.8806$

Equation of the line of regression of Y on X is

$Y - \overline{Y} \qquad = b_{YX}(X - \overline{X})$

$Y - 38 \qquad = -0.6643\ (x - 32)$

$Y \qquad = -0.6643\ x + 38 + 0.6643 \times 32$

$\qquad = -0.6642\ x + 59.2576$

Coefficient of Correlation

$r^2 \qquad = b_{YX} \times b_{XY}$

$\qquad = -0.6643 \times (-0.2337)$

$r \qquad = 0.1552$

$r \qquad = \pm\sqrt{0.1552}$

$r \qquad = \pm\sqrt{0.394}$

Now we have to find the most likely mark, in statistics (Y) when marks in economics (X) are 30.

$y \qquad = -0.6643\ x + 59.2575$

Put x = 30, we get

$y = -0.6643 \times 30 + 59.2536$

$\quad = 39.3286$

$y \simeq 39$

2.4 COVARIANCE

Def : If X and Y are random variables, then Covariance between X and Y is defined as

$\text{Cov}(X, Y) \qquad = E(XY) - E(X) . E(Y)$

$\text{Cov}(X, Y) \qquad = 0$ [If X & Y are independent]

2.4.1 CORRELATION

Types of Correlation

- Positive Correlation
  (If two variables deviate in same direction)
- Negative Correlation
  (If two variables constantly deviate in opposite direction)

2.4.2 KARL-PEARSON'S COEFFICIENT OF CORRELATION

Correlation coefficient between two random variables X and Y usually denoted by r(X, Y) is a numerical measure of linear relationship between them and is defined as

$$r(X, Y) \qquad = \frac{\text{Cov}(X, Y)}{\sigma_X . \sigma_Y},$$

Where $\text{Cov}(X, Y) = \frac{1}{n}\sum XY - \overline{X}\ \overline{Y}$

$$\sigma_X = \frac{\sum X}{n}; \quad \sigma_Y = \frac{\sum Y}{n}$$

* Limits of correlation coefficient

$-1 \le r \le 1.$

X & Y independent, $\therefore$ r(X, Y) = 0.

Note : Types of correlation based on 'r'.

Values of 'r'	Correlation is said to be
r = 1	perfect and positive
0<r<1	positive
-1<r<0	negative
r = 0	Uncorrelated

SOLVED PROBLEMS ON CORRELATION

Example 2.4.1

Calculated the correlation coefficient for the following heights of fathers X and their sons Y.

	5	6	7	7	8	9	0	2
	7	8	5	8	2	2	9	1

Solution

		U = X – 68	**V = Y – 68**	**UV**	**U^2**	**V^2**
5	7	-3	-1	3	9	1
6	8	-2	0	0	4	0
7	5	-1	-3	3	1	9
7	8	-1	0	0	1	0
8	2	0	4	0	0	16
9	2	1	4	4	1	16
0	9	2	1	2	4	1
2	1	4	3	12	16	9

$$\sum U = 0 \qquad \sum V = 0 \qquad \sum UV = 24 \; \sum U^2 = 36 \; \sum V^2 = 52$$

Now

$$\overline{U} = \frac{\sum U}{n} = \frac{0}{8} = 0$$

$$\overline{V} = \frac{\sum V}{n} = \frac{8}{8} = 1$$

Cov (X, Y) = Cov (U, V)

$$\Rightarrow \frac{\sum UV}{n} - \overline{U}\,\overline{V} = \frac{24}{8} - 0 = 3 \qquad (1)$$

$$\sigma_U = \sqrt{\frac{\sum U^2}{n} - \overline{U}^2} = \sqrt{\frac{36}{8} - 0} = 2.121 \qquad (2)$$

$$\sigma_V = \sqrt{\frac{\sum V^2}{n} - \overline{V}^2} = \sqrt{\frac{52}{8} - 1} = 2.345 \qquad (3)$$

$$\therefore r(X, Y) = r(U, V) = \frac{Cov(U, V)}{\sigma_U . \sigma_V} = \frac{3}{2.121 \text{ x } 2.345}$$

= 0.6031 (by 1, 2, 3)

Example :2.4.2

Let X be a random variable with p.d.f. $f(x) = \frac{1}{2}, -1 \le x \le 1$ and let $Y = x^2$, find the correlation coefficient between X and Y.

Solution

$$E(X) = \int_{-\infty}^{\infty} x.f(x)\, dx$$

$$= \int_{-1}^{1} x.\frac{1}{2}\, dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

E(X) = 0

$$E(Y) = \int_{-\infty}^{\infty} x^2.f(x)\, dx$$

$$= \int_{-1}^{1} x^2.\frac{1}{2}\, dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{1}{2}.\frac{2}{3} = \frac{1}{3}$$

$E(XY) = E(XX^2)$

$$= E(X^3) = \int_{-\infty}^{\infty} x^3.f(x)\, dx = \left(\frac{x^4}{4}\right)_{-1}^{1} = 0$$

E(XY) = 0

$$\therefore r(X, Y) = \rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = 0$$

$\rho = 0.$

Note : E(X) and E(XY) are equal to zero, noted not find $\sigma_x \& \sigma_y$.

2.5 TRANSFORMS OF TWO DIMENSIONAL RANDOM VARIABLE

Formula:

$$f_U(u) = \int_{-\infty}^{\infty} f_{u,v}(u,v)\,dv$$

$$\& \qquad f_V(u) = \int_{-\infty}^{\infty} f_{u,v}(u,v)\,du$$

$$f_{UV}(u,V) = f_{XY}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

Example : 2.5.1

If the joint pdf of (X, Y) is given by $f_{xy}(x, y) = x+y$, $0 \le x, y \le 1$, find the pdf of $\cup = XY$.

Solution

Given $\quad f_{xy}(x, y) = x + y$

Given $\quad U = XY$

Let $\quad V = Y$

$$x = \frac{u}{v} \;\&\; y = V$$

$$\frac{\partial x}{\partial u} = \frac{1}{V}.\frac{\partial x}{\partial v} = \frac{-u}{V^2};\frac{\partial y}{\partial u} = 0;\ \frac{\partial y}{\partial v} = 1 \qquad (1)$$

$$\therefore J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{V} & \frac{-u}{V^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v} - 1 = \frac{1}{v}$$

$$\Rightarrow |J| = \frac{1}{V} \qquad (2)$$

The joint p.d.f. (u, v) is given by

$$f_{uv}(u,v) = f_{xy}(x,y)\,|J|$$

$$= (x+y)\frac{1}{|v|}$$

$$= \frac{1}{V}\left(\frac{u}{v} + u\right) \qquad (3)$$

The range of V :

Since $0 \le y \le 1$, we have $0 \le V \le 1$ $\quad (\therefore\ V = y)$

The range of u :

Given $\quad 0 \le x \le 1$

$$\Rightarrow \qquad 0 \le \frac{u}{v} \le$$

$\Rightarrow \qquad 0 \le u \le v$

Hence the p.d.f. of (u, v) is given by

$$f_{uv}(u,v) = \frac{1}{v}\left(\frac{u}{v} + v\right),\ 0 \le u \le v,\ 0 \le v \le 1$$

Now

$$f_U(u) = \int_{-\infty}^{\infty} f_{u,v}(u,v)\,dv$$

$$= \int_u^1 f_{u,v}(u,v)\,dv$$

$$= \int_u^1 \left(\frac{u}{v^2} + 1\right) dv$$

$$= \left[v + u.\frac{v^{-1}}{-1} \right]_u^1$$

$\therefore f_u(u) = 2(1-u),\ 0 < u < 1$

p.d.f of (u, v)	p.d.f of u = XY
$f_{uv}(u,v) = \frac{1}{v}\left(\frac{u}{v} + v\right)$	$f_u(u) = 2(1-u),\ 0 < u < 1$
$0 \le u \le v,\ 0 \le v \le 1$	

TUTORIAL QUESTIONS

1. The jpdf of r.v X and Y is given by f(x,y)=3(x+y),0<x<1,0<y<1,x+y<1 and 0 otherwise. Find the marginal pdf of X and Y and ii) Cov(X,Y).

2. Obtain the correlation coefficient for the following data:

X:	68	64	75	50	64	80	75	40	55	64
Y:	62	58	68	45	81	60	48	48	50	70

3.The two lines of regression are 8X-10Y+66=0, 4X-18Y-214=0.The variance of x is 9 find i) The mean value of x and y. ii) Correlation coefficient between x and y.

4. If $X_1, X_2, \ldots X_n$ are Poisson variates with parameter $\lambda=2$, use the central limit theorem to find $P(120 \le S_n \le 160)$ where $Sn = X_1 + X_2 + \ldots X_n$ and n=75.

5. If the joint probability density function of a two dimensional random variable (X,Y) is

given by f(x, y) = x^2 +—, 0<x<1,0<y<2= 0, elsewhere Find (i) P(X>1/2)(ii) P(Y<X) and (iii) P(Y<1/2/ X<1/2).

6. Two random variables X and Y have joint density

Find Cov (X,Y).

7. If the equations of the two lines of regression of y on x and x on y are respectively

7x-16y+9=0; 5y-4x-3=0, calculate the coefficient of correlation.

WORKEDOUT EXAMPLES

Example 1
The j.d.f of the random variables X and Y is given

$$f(x,y) = \begin{cases} 8xy, & 0 < x < 1, \quad 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

Find (i) $f_X(x)$ (ii) $f_Y(y)$ (iii) $f(y/x)$

Solution
We know that
(i) The marginal pdf of 'X' is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 8xy\,dy = 4x^3$$

$$f(x) = 4x^3,\ 0 < x < 1$$

(ii) The marginal pdf of 'Y' is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 8xy\,dy = 4y$$

$$f(y) = 4y,\ 0 < y < \alpha$$

(iii) We know that

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{8xy}{4x^3} = \frac{2y}{x^2},\ 0 < y < x,\ 0 < x < 1$$

Example 2

Let X be a random variable with p.d.f. $f(x) = \frac{1}{2}, -1 \le x \le 1$ and let $Y = x^2$, find the correlation coefficient between X and Y.

Solution

$$E(X) = \int_{-\infty}^{\infty} x.f(x)\,dx$$

$$= \int_{-1}^{1} x.\frac{1}{2}\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

$$E(X) = 0$$

$$E(Y) = \int_{-\infty}^{\infty} x^2.f(x)\,dx$$

$$= \int_{-1}^{1} x^2.\frac{1}{2}\,dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{1}{2}.\frac{2}{3} = \frac{1}{3}$$

$$E(XY) = E(XX^2)$$

$$= E(X^3) \qquad = \int_{-\infty}^{\infty} x^3.f(x)\,dx = \left(\frac{x^4}{4}\right)_{-1}^{1} = 0$$

$E(XY) = 0$

$$\therefore r(X, Y) = \rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = 0$$

$\rho = 0.$

Note : E(X) and E(XY) are equal to zero, noted not find $\sigma_x \& \sigma_y$.

Result

Marginal pdf g	**Marginal pdf y**	**F(y/x)**
$4x^3$, 0<x<1	4y, 0<y<x	$\frac{2y}{x^2}, 0 < y < x, 0 <$

UNIT – III

RANDOM PROCESSES

Introduction

In chapter 1, we discussed about random variables. Random variable is a function of the possible outcomes of a experiment. But, it does not include the concept of time. In the real situations, we come across so many time varying functions which are random in nature. In electrical and electronics engineering, we studied about signals.

Generally, signals are classified into two types.

(i) Deterministic

(ii) Random

Here both deterministic and random signals are functions of time. Hence it is possible for us to determine the value of a signal at any given time. But this is not possible in the case of a random signal, since uncertainty of some element is always associated with it. The probability model used for characterizing a random signal is called a random process or stochastic process.

3.1 Random Process (Definition):

A random process is a collection of random variables {X(s,t)} that are fxy of a real variable, namely time t where $s \in S$ and $t \in T$ (S: Sample space θ t parameters set or under set).

3.2 Classification of Random Process

We shall consider only four cases based on t.

1) Continuous Random Process.
2) Continuous Random Sequence.
3) Discrete Random Process.
4) Discrete Random Sequence.

Statistical (Ensemble) Averages:

1. Mean = $E\left[X(t)\right] = \int_{-\infty}^{\infty} xf\left(x,t\right)dx$

2. Auto correlation fy of [x(t)]

$$R_{XX}\left(t_1,t_2\right) = E\left[X\left(t_1\right)X\left(t_2\right)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2f\left(x_1,x_2,t_1,t_2\right)dx_1dx_2$$

(or)

$$R_{XX}\left(t_1+t_2+\tau\right) = E\left[X\left(t\right)X\left(t+\tau\right)\right]$$

When τ = time difference = $t_2 - t_1$

3) Auto covariance of [X(t)]

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2).E\left[X(t_1)EX(t_2)\right]$$

$$C_{XX}(t,t) = E\left[X^2(t)\right] - E\left[X(t)\right]^2 \qquad [\because t_1 = t_2 = t]$$

$$= \text{Var}\,[X(t)]$$

4) Correlation coefficient of [X(t)]

$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_2) C_{XX}(t_2, t_3)}}$$

Note : $\rho_{XX}(t, t) = 1$

5) Cross correlation

$$R_{XY}(t_1, t_2) = E\left[X(t_1)Y(t_2)\right]$$

(or)

$$R_{XY}(t, t+\tau) = E\left[X(t)Y(t+\tau)\right]$$

6) Cross Covariance

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - E\left[X(t_1)\right]E\left[Y(t_2)\right]$$

Or $$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E\left[X(t)\right]E\left[Y(t+\tau)\right]$$

Cross Correlation Coefficient

$$\rho_{XY}(t_1, t_2) = \frac{C_{XY}(t_1, t_2)}{\sqrt{C_{XY}(t_1, t_1) C_{XY}(t_2, t_2)}}$$

3.3 FIRST ORDER STRICTLY STATIONARY PROCESS

Stationary Process (or) Strictly Stationary Process (or) Strict Sense Stationary Process [SSS Process]

A random process X(t) is said to be stationary in the strict sense, it its statistical characteristics do not change with time.

Stationary Process:

Formula: $$E\left[X(t)\right] = \text{Constant}$$

$$\gamma\left[X(t)\right] = \text{Constant}$$

1) Consider the RP X(t) = Cos (w_0t +θ) where θ is uniformly distributed in the interval -π to π. Check whether X(t) is stationary or not? Find the first and Second moments of the process.

Given X(t) = cos (W_0t + θ)

Where θ is uniformly distributed in (-π, π)

$$f(\theta) = \frac{1}{\pi - (-\pi)} - \frac{1}{2\pi}, \qquad -\pi < 0 < \pi \qquad \text{[from the def. of uniform distribution]}$$

To prove (i) X(t) is SSS process

(ii) E[X(t)] = Constant

(iii) Var [X(t)] = Constant

$$E\left[X(t)\right] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$$

$$= \int_{-\pi}^{\pi} \cos(w_0 t + \theta) . \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi}\left[\sin(w_0 t + \theta)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\sin w_0 t + \pi\right] + \sin\left[\pi - w_0 t\right]$$

$$= \frac{1}{2\pi}\left[-\sin w_0 t + \sin w_0 t\right] = 0$$

$$E\left[X^2(t)\right] = E\left[ws^2(w_0 t + \theta)\right]$$

$$= \frac{1}{2} E\left[1 + \cos(2w_0 t + 2\theta)\right]$$

$$E[1] = \int_{-\pi}^{\pi} \frac{1}{2\pi} d\theta = 1$$

$$E\left[\cos(2w_0 t + 2\theta)\right] = \int_{-\pi}^{\pi} \cos(2w_0 t + 2\theta) . \frac{1}{2}\pi$$

$$= \frac{1}{2\pi}\left[\sin\frac{(2\omega_0 t + 2\theta)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}[0] = 0$$

$$\therefore \Rightarrow E\left[X^2(t)\right] = \frac{1}{2}(1) + 0 = Y_2$$

$$\text{Var}\left[X(t)\right] = E\left[X^2(t)\right] - \left[E\left[X(t)\right]\right]^2$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2} = \text{const}$$

$\therefore$ X(t) is a SSS Process./

S.T the RP X(t): $A\cos(w_0 t + \theta)$ is not stationary if A and w_0 are constants and θ is uniformly distributed random variable in $(0, \pi)$.

In X(t) = $A\cos(w_0 t + \theta)$

In 'θ' uniformly distributed

$$f(\theta) = \frac{1}{\pi - 0} = \frac{1}{\pi} \qquad 0 < \theta < \pi$$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$$

$$= \int_{0}^{\pi} A\cos(w_0 t + \theta) \, {}^{1}\!/_{\pi} d\theta$$

$$= \frac{A}{\pi}\left[\sin(w_0 t + \theta)\right]_0^{\pi}$$

$$= -\frac{2A}{\pi}\sin w_0 t \neq \text{const.}$$

$\therefore$ N(t) is not a stationary process.

3.3.1 SECOND ORDER AND WIDE SENSE STATIONARY PROCESS

A Process is said to be second order stationary, if the second order density function statistics.

$$f(x_1, x_2 : t_1, t_2) = f(x_1, x_2 : t + \delta, t_2 + \delta), \ \forall x_1 \, x_2 \text{ and } \delta$$

If a random process X(t) is WSS then it must also be covariance stationary.

In X(t) is WSS

i) E[X(t)] = μ = a const.

(ii) $R(t_1, t_2)$ = a fy of $(t_1 - t_2)$

The auto covariance fn is gn by

$$C(t_1, t_2) = R(t_1, t_2) - E[X(t_1)X(t_2)]$$

$$= R(t_1 - t_2) - E[X(t_1)]E[X(t_2)]$$

$$= R(t_1 - t_2) - \mu(\mu)$$

$$= R(t_1 - t_2) - \mu^2$$

Which depends only on the time difference. Hence X(t) is covariance stationary.

If X(t) is a wide sense stationary process with auto correlation $R(\tau) = Ae^{-\alpha(\tau)}$, determine the second order moment of the random variable X(8) - X(5).

Given $R(\tau) = Ae^{-\alpha(\tau)}$

$$R(t_1, t_2) = Ae^{-\alpha(t_1 - t_2)}$$

$$E[X^2(t)] = R(t, t) = A$$

$$E[X^2(8)] = A$$

$$E[X^2(5)] = A$$

$$E[X(8)X(15)] = R|8,5| = Ae^{-\alpha}(8,5)$$

$$= Ae^{-3\alpha}$$

The second moment of X(8) - X(5) is given by

$$E[X(8) - X(15)]^2 = E[X^2(8)]^2 + E[X^2(5)] - 2E[X(8)X(5)]$$

$= A + A - 2Ae^{-3\alpha}$
$= 2A(1 - e^{-3\alpha})$

Example:3.3.1

S.T the random process $X(t) = A\cos(w_t+\theta)$ is wide sense stationary if A & w are constants and θ is uniformly distributed random variable in $(0, 2\pi)$.

Given $X(t) = A\cos(wt + \theta)$

$$f(\theta) = \frac{1}{2\pi - 0} = \begin{cases} \frac{1}{2\pi}, & 0 < \theta < 2\pi \\ 0 & ,\text{otherwise} \end{cases}$$

$$f(\theta) = 1/2\pi$$

To prove X(t) is WSS

(i) $E[X(t)] = \text{Constant}$

(ii) $R(t_1, t_2) = \text{a fn of } (t_1 - t_2)$

(i) $E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta) d\theta$

$$\Rightarrow E[A\cos(wt+\theta)] = \int_{-\infty}^{\infty} A\cos(wt+\theta) f(\theta) d\theta$$

$= \text{constant}$

(ii) $R(t_1, t_2) = E[X(t_1)x(t_2)]$

$= E[A\cos(wt_1\theta)A\cos(wt_2+\theta)]$

$= E[A^2\cos(wt_1+\theta)\cos(wt_2+\theta)]$

$= \frac{A^2}{2}E[\cos w(t_1+t_2)+2\theta] + \frac{A^2}{2}\cos[w(t_1-t_2)]$

$= 0$

$$\Rightarrow R(t_1, t_2) = \frac{A^2}{2}\cos[w(t_1 - t_2)]$$

$= \text{a fn of } (t_1 - t_2)$

Hence X(t) is a WSS Process.

Example3.3.2

If $X(t) = A\cos\lambda t + B\sin\lambda t$, where A and B are two independent normal random variable with $E(A) = E(B) = 0$, $E(A^2) = E(B^2) = \sigma^2$, where λ is a constant. Prove that $\{X(t)\}$ is a Strict Sense Stationary Process of order 2 (or)

If $X(t) = A\cos\lambda t + B\sin\lambda t$, $t \geq 0$ is a random process where A and B are independent $N(0, \sigma^2)$ random variable, Examine the WSS process X(t).

Given $X(t) : A\cos\lambda t + B\sin\lambda t$ --------------- (1)

$E(A) = 0;$ $E(B) = 0$ --------------- (2)

$E(A^2) = \sigma^2 = k;\ E(B^2) = \sigma^2 = k$

$E[AB] = E[A]E[B]$ $\quad [\because A \ \&\ B \text{ are independent}]$

$= 0$

To prove : X(t) is WSS Process

i.e. (i) $E[X(t)] = \text{Constant}$

(ii) $R(t_1, t_2) = \text{a fn of } (t_1 - t_2)$

$E[X(t)] \quad = E[A\cos\lambda t + B\sin\lambda t]$

$= \cos\lambda t E[A] \sin\lambda t E(B)$

$= 0$

$R(t_1, t_2) = E[X(t_1)X(t_2)]$

$= E[(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)]$

$= E[A^2\cos\lambda t_1\cos\lambda t_2 + B^2\sin\lambda t_1\sin\lambda t_2 +$

$AB\cos\lambda t_1\sin\lambda t_2 + \sin\lambda t_1\cos\lambda t_2]$

$= \cos\lambda t_1\cos\lambda t_2 E[A^2] + \sin\lambda t_1\sin\lambda t_2 E[B^2]$

$+ E[AB][\sin(\lambda t_1 + \lambda t_2)]$

$= K\cos\lambda t_1\cos\lambda t_2 + K\sin\lambda t_1\sin\lambda t_2 + 0$

$= K\cos\lambda(t_1 - t_2)$

$= \text{a fn of } (t_1 - t_2)$

Both the conditions are satisfied. Hence X(t) is a WSS Process.

Example:3.3.3

Consider a random process X(t) defined by $N(t) = U\cos t + (V+1)\sin t$, where U and V are independent random variables for which E(U) = E(V) = 0, $E(U^2) = E(V^2) = 1$. Is X(t) is WSS? Explain your answer?

Given $X(t) = U\cos t + (v+1)\sin t$

E(U) = E(V) = 0

$E(U^2) = E(V^2) = 1$

E(UV) = E(U) E(V) = 0

$E[X(t)] \quad = E[V\cos t + (V+1)\sin t]$

$= E(U)\cos t + E(V)\sin t + \sin t$

= 0 + 0 + sin t

= sin t

$\neq$a constant

$\Rightarrow X(t)$ is not a WSS Process.

3.3.2 ERGODIC PROCESS

Ergodic Process are processes for which time and ensemble (statistical) averages are interchangeable the concept of ergodicity deals with the equality of time and statistical average.

Time Average

It X(t) is a random process, then $\frac{1}{2\pi}\int_{-T}^{T} X(t)\,dt$ is called the time average of X(t) over (-T, T) and is denoted by $\overline{X}_T$.

Note

1. $\overline{X}_T = \frac{1}{2\pi}\int_{-T}^{T} X(t)\,dt,$ X(t) defined in (–T, T)
2. $\overline{X}_T = \frac{1}{T}\int_{0}^{T} X(t)\,dt,$ D(t) defined in (0, T)

3.4 MARKOV PROCESS - MARKOV CHAINS

3.4.1 Markov Process

A random process X(t) is said to be Markov Process, it

$$P\left[X(t) \le x / X(t_1) - x_1, X(t_2) = x_2 \ldots X(t_n) = x_n\right]$$

$$= P\left[X(t) \le x / X(t_n) = x_n\right]$$

3.4.2 Markov Chain

A discrete parameter Markov Process is called Markov Chain.

If the tmp of a Markov Chain is $\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, find the steady state distribution of the chain.

Given $P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

If $\pi = (\pi_1 \pi_2)$ is the steady state distribution of the chain, then by the property of π, we have

$$\pi P = \pi \qquad ----------(1)$$

$$\pi_1 + \pi_2 = 1 \qquad ----------(2)$$

$$\Rightarrow (\pi_1\, \pi_2)\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = (\pi_1 \pi_2)$$

$$\left[\pi_1[0] + \pi_2\left(\tfrac{1}{2}\right) + \pi_1(1) + \pi_2\left(\tfrac{1}{2}\right)\right] = [\pi_1\ \pi_2]$$

$\frac{1}{2}\pi_2 = \pi_1 \qquad ----------- (3)$

$\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \qquad --------- (4)$

$\pi_1 + \pi_2 = 1$

$\frac{1}{2}\pi_2 + \pi_2 = 1 \Rightarrow \frac{3}{2}\pi_2 = 1$ by (3)

$\pi_2 = \frac{2}{3}$

(3) $\Rightarrow \pi_1 = \frac{1}{2}\pi_2 = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$

$\therefore$ The steady state distribution of the chain is $\pi = (\pi_1\ \pi_2)$

i.e. $\pi = \left(\frac{1}{3}\ \frac{2}{3}\right)$

Example :3.4.1

An Engineering analysing a series of digital signals generated by a testing system observes that only 1 out of 15 highly distorted signals followed a highly distorted signal with no recognizable signal, where as 20 out of 23 recognizable signals follow recognizable signals with no highly distorted signals b/w. Given that only highly distorted signals are not recognizable. Find the fraction of signals that are highly distorted.

π_1 = The fraction of signals that are recognizable [R]

π_2 = The fraction of signals that are highly distorted [D]

The tmp of the Markov Chain is

$$P = \begin{array}{c} \\ R \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cc} R & D \end{array} \\ \begin{bmatrix} \frac{20}{23} & - \\ - & \frac{1}{15} \end{bmatrix} \end{array} \qquad \Rightarrow P = \begin{bmatrix} \frac{20}{23} & \frac{3}{23} \\ \frac{14}{15} & \frac{1}{15} \end{bmatrix}$$

- 1 out of 15 highly distorted signals followed a highly distorted signal with no recognizable signal

$[P(D \to D)] = \frac{1}{15}$

- 20 out of 23 recognizable signals follow recognizable signals with no highly distorted signals.
- If the tmp of a chain is a stochastic martin, then the sum of all elements of any row is equal to 1.

If $\pi = (\pi_1 \pi_2)$ is the steady state distribution of the chain, then by the property of π, we have

$\pi P = \pi$

$\pi_1 + \pi_2 = 1$

$$\Rightarrow (\pi_1 \pi_2)\begin{bmatrix} 20/23 & 3/23 \\ 14/15 & 1/15 \end{bmatrix} = (\pi_1 \pi_2)$$

$$20/23\, \pi_1 + 14/15\, \pi_2 = \pi_1 \qquad -----(3)$$

$$3/23\, \pi_1 + 1/15\, \pi_2 = \pi_2 \qquad -----(4)$$

(3) $\quad \pi_2 = 45/322\, \pi_1$

$\Rightarrow \pi_1 = 322/367$

(2)

$\pi_2 = 45/367$

$\therefore$ The steady state distribution of the chain is

$$\pi = (\pi_1 \pi_2)$$

i.e. $\pi = \left(\frac{322}{367} \quad \frac{45}{367}\right)$

$\therefore$ The fraction of signals that are highly distorted is $45/367$

$$\Rightarrow \frac{45}{367} \text{x}100\% = 12.26\%$$

Example :3.4.2

Transition prob and limiting distribution. A housewife buys the same cereal in successive weeks. If the buys cereal A, the next week she buys cereal B. However if she buys B or C, the next week she is 3 times as likely to buy A as the other cereal. In the long run how often she buys each of the 3 cereals.

Given : Let $\pi_1 \to$ Cereal A

$\pi_2 \to$ Cereal B

$\pi_3 \to$ Cereal C

$\therefore$ the tpm of the Markov Chain is

$$P = \begin{bmatrix} 0 & 1 & - \\ 3/4 & 0 & - \\ 3/4 & - & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix}$$

If $\pi = (\pi_1, \pi_2, \pi_3)$ is the steady - state distribution of the chain then by the property of π we have,

$$\pi P = \pi$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\Rightarrow (\pi_1\,\pi_2\,\pi_3)\begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} = [\pi_1\,\pi_2\,\pi_3]$$

$$\tfrac{3}{4}\pi_2 + \tfrac{3}{4}\pi_3 = \pi_1 \qquad (3)$$

$$\pi_1 + \tfrac{1}{4}\pi_3 = \pi_2 \qquad (4)$$

$$\tfrac{1}{4}\pi_2 = \pi_3 \qquad (5)$$

$$(3) \Rightarrow \tfrac{3}{4}\pi_2 + \tfrac{3}{4}\left(\tfrac{1}{4}\pi_2\right) = \pi_1 \qquad \text{(by 5)}$$

$$\tfrac{15}{16}\,\pi_2 = \pi_2 \qquad (6)$$

$$(2) \Rightarrow \pi_1 + \pi_2 + \pi_3 = 1$$

$$\tfrac{15}{16}\pi_2 + \pi_2 + \frac{1}{4}\pi_2 = 1 \qquad \text{by (5) \& (6)}$$

$$\frac{35}{16}\pi_2 = 1$$

$$\pi_2 = \frac{16}{35}$$

$$(6) \Rightarrow \pi_1 = \frac{15}{16}\pi_2$$

$$\pi_1 = \frac{15}{35}$$

$$(5) \Rightarrow \pi_3 = \frac{1}{4}\pi_2$$

$$\pi_3 = \frac{4}{35}$$

$\therefore$ The steady state distribution of the chain is

$$\pi = (\pi_1\,\pi_2\,\pi_3)$$

$$\text{i.e., } \pi = \left(\tfrac{15}{35} \quad \tfrac{16}{35} \quad \tfrac{4}{35}\right)$$

n - step tmp P^n

Example : 3.4.1 The transition Prob. Martix of the Markov Chain $\{X_n\}$, n = 1,2,3,… having 3 states 1, 2 & 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $P^{(0)} = (0.7 \quad 0.2 \quad 0.1)$.

Find (i) $P(X_2 = 3)$ and (ii) $P\left[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\right]$

Solution

Given $P^{(0)} = (0.7 \quad 0.2 \quad 0.1)$.

$$\Rightarrow \qquad P\left[X_0 = 1\right] = 0.7$$

$$P\left(X_0 = 2\right) = 0.2$$

$$P\left[X_0 = 3\right] = 0.1$$

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}^{(1)} & P_{12}^{(1)} & P_{13}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & P_{23}^{(1)} \\ P_{31}^{(1)} & P_{32}^{(1)} & P_{33}^{(1)} \end{bmatrix}$$

$$P^2 = P \,.\, P$$

$$= \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix}$$

(i) $P\left[X_2 = 3\right] = \sum_{i=1}^{3} P\left[X_2 = 3 / X_0 = i\right] P\left[\lambda_0 = i\right]$

$$= P\left[X_2 = 3 / X_0 = 1\right] P\left[X_0 = 1\right] + P\left[X_2 = 3 / X_0 = 3\right] P\left[X_0 = 2\right] +$$

$$P\left[X_2 = 3 / X_{0=3}\right] P\left[X_0 = 3\right]$$

$$= P_{13}^{(2)} P\left[X_0 = 1\right] + P_{23}^{(2)} P\left[X_0 = 2\right] + P_{33}^{(2)} P\left[X_0 = 3\right]$$

$$= (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1)$$

$$= 0.279$$

(ii) $P\left[X_3 = 2, \;\; X_2 = 3, \;\; X_1 = 3, \;\; X_0 = 2\right]$

$$= P_{32}^{(1)} P\left[X_2 = 3 / X_1 = 3, X_0 = 2\right] P\left[\lambda_1 = 3, X_0 = 2\right]$$

$$= P_{32}^{(1)} \; P_{33}^{(1)} \; P_{23}^{(1)} \; P\left[X_0 = 2\right]$$

$= (0.4)\ (0.3)\ (0.2)\ (0.2)$
$= 0.0048$

3.5 TYPE 5

A training process is considered as two State Markov Chain. If it rain, it is considered to be state 0 & if it does not rain the chain is in stable 1. The tmp of the Markov Chain is defined as

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

i. Find the Prob. That it will rain for 3 days from today assuming that it is raining today.

ii. Find also the unconditional prob. That it will rain after 3 days with the initial Prob. Of state) and state 1 as 0.4 & 0.6 respectively.

Solution:

Given $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$

$$P^{(2)} = P^2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}\begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix}$$

$$P^{(3)} = P^3 = P^2 P$$

$$= \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix}$$

(i) If it rains today, then Prob. Distribution for today is (1 0)

$$\therefore P(\text{after 2 days}) = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix}$$

$$= [0.376 \quad 0.624]$$

$\therefore$ P [Rain for after 3 days] $= 0.376$

(ii) Given $P^{(0)} = \begin{pmatrix} 0.4 & 0.6 \end{pmatrix}$

P[after 3 days] $= \begin{pmatrix} 0.4 & 0.6 \end{pmatrix}\begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix}$

$= \begin{pmatrix} 0.3376 & 0.6624 \end{pmatrix}$

$\therefore$ P[rain for after 3 days] $= 0.3376$

Example :3.5.1

Prove that the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ is the tpm of an irreducible Markov Chain?

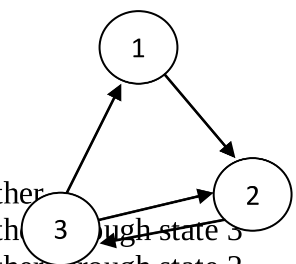
(or)

Three boys A, B, C are throwing a ball each other. A always throws the ball to B & B always throws the ball to C but C is just as like to throw the ball to B as to A. State that the process is Markov Chain. Find the tpm and classify the status.

The tpm of the given Markov chain is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(a) let $X_n = \{1, 2, 3\} \Rightarrow$ finite



State 2 & 3 are communicate with each other

State 1 & 2 are communicate with each other through state 3

State 3 & 1 are communicate with each other through state 2.

$\Rightarrow$ The Markov Chain is irreducible (3)

From (1) & (2) all the states are persistent and non-null (3)

One can get back to

State 1 in $1 \to 2 \to 3 \to 1$ (3 steps)

State 2 in $2 \to 3 \to 2$ (2 steps)

State 2 in $2 \to 3 \to 1 \to 2$ (3 steps)

State 3 in $3 \to 1 \to 2 \to 3$ (3 steps)

State 3 in $3 \to 2 \to 2$ (2 steps)

$\Rightarrow$ The states are aperiodic (4)

[$\because$ The states are not periodc]

From (3) & (4) we get all the states are Ergodic.

3.6 POISSON BINOMIAL PROCESS

If X(t) represents the no. of occurrences of certain even in (0, t), then the discrete random process {X(t)} is called the Poisson process, provided the following postulate are satisfied.

i. P[1 occurrence in $(t, t+\Delta t) = \lambda\Delta t$

ii. P[0 occurrence in $(t, t+\Delta t) = 1 - \lambda\Delta t$

iii. P[2 occurrence in $(t, t+\Delta t) = 0$

iv. X(t) is independent of the no. of occurrences of the event in any interval prior and after the interval (0, t)

v. The Prob. That the events occurs a specified no. of times in (t_0, t_0+t) depends only on t, but not on t_0.

Prob. Law for the Poisson Process {X(t)}

$$P\left[X(t)=n\right]=P_n(t)=\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$n = 0, 1, 2, 3, \ldots.$$

3.7 BINOMIAL PROCESS

Let X_n, n = 1, 2, 3, … be a Bernoulli Process and S_n denote the No. of the successes in the 1st n Bernoulli trails i.e., $S_n = X_1 + X_2 + \ldots + X_n$ $P\left[X_n = k\right] = \binom{n}{k} P^k q^{n-k}$, k = 0, 1, 2, …..

Example:3.7.1

Suppose that customers arrive at a bank according to a Poisson Process with mean rate of 3 per minute. Find the Prob. That during a time interval of 2 minutes (i) exactly 4 customer arrive(ii)Greater than 4 customer arrive (iii) Fewer than 4 customer arrive.

$\lambda = 3$

$$P\left[X(t)=n\right] = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0,1,2\ldots$$

$$= \frac{E^{-3t}(3t)^n}{n!} \qquad n = 0,1,\ldots$$

P (Exactly 4 customers in 2 minutes)

$$= P\left[X(2)=4\right] = \frac{e^{-664}}{4!} = 0.1338$$

(ii) P[more than 4 customers in 2 minutes]

$$= P\left[X(2)>4\right] = 1 - P\left[X(2)\le 4\right]$$

$$= 1 - e^{-6}\left[1 + 6 + 62\big/_{2!} + 63\big/_{3!} + 64\big/_{4!}\right]$$

$$= 0.7149$$

(iii) P[Fewer than 4 customer arrive in 2 minutes]

$$= P\left[X(2)<4\right]$$

$$= e^{-6}\left[1 + 6 + 62\big/_{2!} + 63\big/_{3!}\right]$$

$$= 0.1512$$

Example:3.7.2

If customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the Prob. that the interval 6/w two consecutive arrivals is (i) more than 1 minute (ii) B/W 1 & 2 minute (iii) 4 minutes or less

$\lambda = 2$

(i) $P[T > 1] = \int_1^{\infty} 2e^{-2t} dt = 0.1353$

(ii) $P[1 < T < 2] = 0.1170$

(iii) $P[T \le 4] = \int_0^4 2e^{-2t} dt = 0.9996$

TUTORIAL PROBLEMS

1.Train.Now suppose thatonthe firstdayof the week,the man tosseda fair dice and drovetoworkif andonlyif a 6appeared.Find(a)the probabilitythathe takes atrainonthe thirddayand(b) theprobabilityhe drives toworkin thelongrun

2.Agambler has Rs.2/- .He bets Rs.1ata time andwins Rs.1with probability½.He stopsplayingif helosesRs.2or wins Rs.4.(a)whatis thetpmof the relatedMarkovchain? (b) What isthe probabilitythat he haslosthis moneyat theend of5 plays?(c) What isthe probabilitythat thesame game lastsmore than7 plays

3.There are2white marblesinurnAand3redmarblesinurnB.At eachstepof theprocess,a marble is selectedfromeach urnandthe 2 marbles selectedareinterchanged.Letthestate aɪof thesystembe thenumber of redmarblesin Aafter i changes.Whatisthe probabilitythatthere are 2redmarblesinurnA?

4.Threeboys A,B andCare throwinga balltoeachother.Aalways throwsthe balltoB andB always throwsthe balltoC, butCisjustas likelyto throwtheballtoB as toA.Showthattheprocess is Markovian.Findthetransitionmatrixandclassifythe states.

WORKED OUT EXAMPLES

Example :1 The transition Prob. Martix of the Markov Chain $\{X_n\}$, n = 1,2,3,… having 3 states 1, 2 & 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $P^{(0)}$= (0.7 0.2 0.1).

Find (i) $P(X_2 = 3)$ and (ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

Solution

Given $P^{(0)}$= (0.7 0.2 0.1).

$$\Rightarrow \quad P\left[X_0 = 1\right] = 0.7$$

$$P\left(X_0 = 2\right) = 0.2$$

$$P\left[X_0 = 3\right] = 0.1$$

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}^{(1)} & P_{12}^{(1)} & P_{13}^{(1)} \\ P_{21}^{(1)} & P_{22}^{(1)} & P_{23}^{(1)} \\ P_{31}^{(1)} & P_{32}^{(1)} & P_{33}^{(1)} \end{bmatrix}$$

$$P^2 = P \,.\, P$$

$$= \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix}$$

(i) $P\left[X_2 = 3\right] = \sum_{i=1}^{3} P\left[X_2 = 3 / X_0 = i\right] P\left[\lambda_0 = i\right]$

$$= P\left[X_2 = 3 / X_0 = 1\right] P\left[X_0 = 1\right] + P\left[X_2 = 3 / X_0 = 3\right] P\left[X_0 = 2\right] +$$

$$P\left[X_2 = 3 / X_{0=3}\right] P\left[X_0 = 3\right]$$

$$= P_{13}^{(2)} P\left[X_0 = 1\right] + P_{23}^{(2)} P\left[X_0 = 2\right] + P_{33}^{(2)} P\left[X_0 = 3\right]$$

$$= (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1)$$

$$= 0.279$$

(ii) $P\left[X_3 = 2, \;\; X_2 = 3, \;\; X_1 = 3, \;\; X_0 = 2\right]$

$$= P_{32}^{(1)} P\left[X_2 = 3 / X_1 = 3, X_0 = 2\right] P\left[\lambda_1 = 3, X_0 = 2\right]$$

$$= P_{32}^{(1)} \; P_{33}^{(1)} \; P_{23}^{(1)} \; P\left[X_0 = 2\right]$$

$$= (0.4)\,(0.3)\,(0.2)\,(0.2)$$

$$= 0.0048$$

UNIT - IV

QUEUEING THEORY

4.1 The input (or Arrival Pattern)

(a) Basic Queueing Process:

Since the customers arrive in a random fashion. Therefore their arrival pattern can be described in terms of Prob. We assume that they arrive according to a Poisson Process i.e., the no f units arriving until any specific time has a Poisson distribution. This is the case where arrivals to the queueing systems occur at random, but at a certain average rate.

(b) Queue (or) Waiting Line

(c) Queue Discipline

It refers to the manner in which the members in a queue are choosen for service

Example:

i. First Come First Served (FIFS)
(or)
First In First Out (FIFO)
ii. Last Come, First Served (LCFS)
iii. Service In Random Order (SIRO)
iv. General Service Discipline (GD)

4.1.1 TRANSIENT STATE

A Queueing system is said to be in transient state when its operating characteristics are dependent on time. A queueing system is in transient system when the Prob. distribution of arrivals waiting time & servicing time of the customers are dependent.

4.1.2 STEADY STATE:

If the operating characteristics become independent of time, the queueing system is said to be in a steady state. Thus a queueing system acquires steady state, when the Prob. distribution of arrivals are independent of time. This state occurs in the long run of the system.

4.2.3 TYPES OF QUEUEING MODELS

There are several types of queueing models. Some of them are

1. Single Queue - Single Server Point
2. Multiple Queue - Multiple Server Point
3. Simple Queue - Multiple Server Point
4. Multiple Queue - Single Server Point

The most common case of queueing models is the single channel waiting line.

Note

P = Traffic intensity or utilization factor which represents the proportion of time the servers are busy = λ/4.

Characteristics of Model I

1) Expected no. of customers in the system is given by

$$\text{Ls (or) E(n)} = \frac{\rho}{1-\rho}$$

$$= \frac{\lambda}{\mu-\mu}$$

2) Expected (or avg) queue length (or) expected no. of customer waiting in the queue is given by

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

3) Expected (or avg.) waiting time of customer in the queue is given by

$$w_q = \frac{L_q}{\lambda}$$

4) Expected (or avg.) waiting time of customer in the system (waiting & service) is given by

$$W_S = \frac{L_S}{\lambda}$$

5) Expected (or avg.) waiting time in the queue for busy system

$$W_b = \frac{\text{Expected waiting time of a customer in the queue}}{\text{Prob.}(\text{System being busy})}$$

(or)

$$W_b = \frac{1}{\mu-\lambda}$$

6) Prob. Of k or more customers in the system

$$P(n \geq k) = \left(\lambda/\mu\right)^k; \quad P(n > k) = \left(\lambda/\mu\right)^{k=1}$$

7) The variance (fluction) of queue length

$$\text{Var}(n) = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

8) Expected no. of customers served per busy

$$\text{Period } L_b = \frac{\mu}{\mu-\lambda}$$

9) Prob. Of arrivals during the service time of any given customer

$$P[X=r] = \left(\frac{\lambda}{\lambda+\mu}\right)^r \left(\frac{\mu}{\lambda+\mu}\right)$$

10) Prob. Density function of waiting time (excluding service) distribution

$$= \lambda\left(1-\lambda/\mu\right)e^{-(\mu+\lambda)t}$$

11) Prob. Density function of waiting + service time distribution

$$= (\mu-\lambda)e^{-(\mu-\lambda)t}$$

12) Prob. Of queue length being greater than on equal to n

$$= \left(\lambda/\mu\right)^{n}$$

13) Avg waiting time in non-empty queue (avg waiting time of an arrival Wh waits)

$$W_n = \frac{1}{\mu-\lambda}$$

14) Avg length of non-empty queue

(Length of queue that is formed from time to time) = $L_n = \frac{\mu}{\mu-\lambda}$

4.3 Littles' Formulae

We observe that $L_S = \lambda W_S$, $L_q = \lambda W_q$ & $W_S = W_q + \frac{1}{\mu}$ and these are called Little's Formulae

4.3.1 Model I: (M/M/1)" (∞/FCFS)

A TV repairman finds that the time spend on his job has an exponential distribution with mean 30 minutes. If the repair sets in the order in which they came in and if the arrival of sets is approximately Poisson with an avg. rate of 10 per 8 hours day, what is the repairman's expected idle time each day? How many jobs are ahead of avg. set just brought?

It is (M/M/1): (∞/FSFC) Problem

Here $\lambda = \frac{10}{8x60} = \frac{1}{48}$ set / minute and

$\mu = \frac{1}{30}$ set /minute

Prob. that there is no unit in the system $P_0 = 1-\frac{\lambda}{\mu}$

$$= 1-5/8 = 3/8$$

Repairman's expected idle time in 8 hours day

$$= nP_0 = 8x\, 3/8 = 3 \text{ hours}$$

Expected avg. no. of jobs (or) Avg. no. of TV sets in the system.

$$L_S = \frac{\lambda}{\mu-\lambda}$$

$$= \frac{1/48}{1/30 - 1/48} = 5/3 \text{ jobs}$$

Example:4.31

In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time (tie time taken to hump to train) distribution is also exponential with an avg. of 36 minutes. Calculate

(i) Expected queue size (line length)

(ii) Prob. that the queue size exceeds 10.

If the input of trains increase to an avg. of 33 per day, what will be the change in (i) & (ii)

$$\lambda = \frac{30}{60x24} = \frac{1}{48} \text{ trains per minute.}$$

$$\mu = 1/36 \text{ trains per minute.}$$

The traffic intensity $\rho = \lambda/\mu$

$= 0.75$

(i) Expected queue size (line length)

$$L_S = \frac{\lambda}{\lambda - \mu} \quad \text{or} \quad \frac{\rho}{1-\rho}$$

$$= \frac{0.75}{1-0.75} = 3 \text{ trains}$$

(ii) Prob. that the queue size exceeds 10

$$P[n \geq 10] = \rho^{10} = (0.75)^{10} = 0.06$$

Now, if the input increases to 33 trains per day, then we have

$$\lambda = \frac{30}{60x24} = \frac{1}{48} \text{ trains per minute.}$$

$$\mu = 1/36 \text{ trains per minute.}$$

The traffic intensity $\rho = \lambda/\mu = \frac{11}{480} x36$

$= 0.83$

Hence, recalculating the value for (i) &(ii)

(i) $L_S = \frac{\rho}{1-\rho} = 5$ trains (approx)

(ii) $P(n \geq 10) = \rho^{10} = (0.83)^{10} = 0.2$ (approx)

Hence recalculating the values for (i) & (ii)

(i) $Ls = \rho / 1\text{-}\rho = 5$ trains (approx)
(ii) $P(n \geq 10) = \rho^{10} = (0.83)^{10} = 0.2$ (approx)

Example:4.3.2

(3) A super market has a single cashier. During the peak hours, customers arrive at a rate of 20 customers per hour. The average no of customers that can be processed by the cashier is 24 per hour. Find

(i) The probability that the cashier is idle.
(ii) The average no of customers in the queue system

(iii) The average time a customer spends in the system.
(iv) The average time a customer spends in queue.
(v) The any time a customer spends in the queue waiting for service

$\lambda = 20$ customers

$\mu = 24$ customers / hour

(i) Prob. That the customer is idle = $1\text{-} \lambda/\mu = 0.167$
(ii) Average no of customers in the system.

$L_s = \lambda / \mu\text{-}\lambda = 5$

(iii) Average time a customer spends in the system

$W_s = L_s / \lambda = ¼$ hour = 15 minutes.

(iv) Average no of customers waiting in the queue

$Lq = \lambda^2 / \mu(\mu\text{-}\lambda) = 4.167$

(v) Average time a customer spends in the queue

$Wq = \lambda / \mu(\mu\text{-}\lambda) = 12.5$ minutes

4.4 Model (IV) : (M/M/I) : (N/FCFS)

Single server finite queue model

$$P_0 = \frac{1\text{-}p}{1\text{-}p^{N+1}}$$ Where $p \neq 1, \rho = \lambda / \mu < 1$

$$P_n = \begin{cases} \dfrac{(1-\rho)}{(1-\rho^{N+1})} p^n, & 0 \le n \le N,\ \rho \ne 1\ (\lambda \ne \mu) \\ \dfrac{1}{N+1}, & \rho = 1 \quad (\lambda = \mu) \end{cases}$$

4.4.3 CHARACTERISTIC OF MODEL IV

(1) Average no of customers in the system

$$Ls = \frac{\lambda}{\mu - \lambda} - \frac{(N+1)\,(\lambda/\mu)^{N+1}}{1 - (\lambda/\mu)^{N+1}}, \qquad \text{if } \lambda \ne \mu$$

$$\text{and} \qquad L_s = \sum_{n=0}^{W} \frac{n}{K+1} = \frac{K}{2}, \qquad \text{if } \lambda = \mu$$

(2) Average no of customers in the queue

$$L_q = L_s - (1 - p_0)$$

$$L_q = L_s - \lambda / \mu, \qquad \text{where } \lambda' = \mu(1 - p_0)$$

(3) Average waiting times in the system and in the queue

$$W_s = L_s / \lambda' \ \&\ W_q = L_q / \lambda', \qquad \lambda' = \mu(1-\rho_0)$$

(4) Consider a single server queuing system with Poisson input, exponential service times, suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours and the maximum permissible no calling units in the system is two. Find the steady state

probability distribution of the no of calling units in the system and the expected no of calling units in the system.

$\lambda = 3$ units per hour

$\lambda = 4$ units per hour & $N = 2$

The traffic intensity $\rho = \lambda/\mu = 0.75$

$$P_n = \frac{(1 - \rho)\rho^n}{1 - \rho^{N+1}}, \rho \neq 1$$

$$= (0.43)\,(0.75)^n$$

$$P_0 = \frac{1 - \rho}{1 - \rho^{N+1}}, = \frac{1 - 0.75}{(1 - 0.75)^{2+1}} \qquad = 0.431$$

The expected no of calling units in the system is equation by

$$Ls = \sum_{n=1}^{W} n\,P_n = 0.81$$

(2) Trains arrive at the yard every 15 minutes and the services time is 33 minutes. If the line capacity of the yard is limited to 5 trains, find the probability that yard is empty and the average no of trains in the system.

$\lambda = 1/15$ per minute; $\mu = 1/33$ per minutes

$\rho = \lambda / \mu = 2.2$

Probability that the yard is empty

$$P_0 = \frac{\rho - 1}{\rho^{N+1} - 1} = \frac{2.2 - 1}{(2.2)^6 - 1} = 1.068\% = 0.01068$$

Average no of trains in the system

$$\begin{aligned} L_s &= \sum_{n=0}^{W} nP_n = P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 \\ &= P_0\left[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5\right] \\ &= 4.22 \end{aligned}$$

4.5 MULTIPLE – CHANNEL QUEUING MODELS

Model V (M / M / C) : (∞ / FCFS)

(Multiple Servers, Unlimited Queue Model)

(M/M/C) (∞/FCFS) Model

$$p_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^n \left(\frac{\lambda C}{\mu C - \lambda}\right)\right]^{-1}$$

Characteristic of this model : (M/M/C) : (∞/FCFS)

(1) $P[n \geq C]$ = Probability that an arrival has to wait (busy period)

$$= \sum_{n=1}^{\infty} P_n = \frac{(\lambda/\mu)^C \mu_C}{C!(C\mu - \lambda)} P_0$$

(2) Probability that an arrival enters the service without wait

$$1 - \frac{C\mu(\lambda/\mu)^C}{C!(C\mu - \lambda)} P_0$$

(3) Average queue length (or) expected no of customers waiting in the queue is

$$L_q = \left[\frac{1}{(C-1)!}(\lambda/\mu)^C \frac{\lambda\mu}{(C\mu - \lambda)^2}\right] P_0$$

(4) Expected no of customers in the system is

$$L_s = L_q + \frac{\lambda}{\mu};\ L_s = \left[\frac{1}{(C-1)!}(\lambda/\mu)^C \frac{\lambda\mu}{(C\mu-\lambda)^2}\right]P_0 + \frac{\lambda}{\mu}$$

(5) Average time an arrival spends in the system

$$W_s = \frac{L_s}{\mu}; = \frac{\mu(\lambda/\mu)^C}{(C-1)!\,(C\mu-\lambda)^2}P_0 + \frac{1}{\mu}$$

(6) Average waiting time of an arrival (expected no of customer spends in the queue for service)

$$W_q = \frac{L_s}{\lambda};\ W_s = \frac{1}{\mu} = \frac{\mu(\lambda/\mu)^C}{(C-1)!\,(C\mu-\lambda)^2}P_0$$

(7) Utilization rate $\rho = \lambda / C\mu$

(8) The probability there are no customers or units in the system is P_0

$$\text{i.e. } P_0 = \left[\sum_{n=0}^{C-1}\frac{1}{n!}(\lambda/\mu)^n + \frac{1}{C!}(\lambda/\mu)^C \frac{C\mu}{(C\mu-\lambda)}\right]^{-1}$$

(9) The probability that there are n units in the system

$$P_n = \begin{cases} \dfrac{\left(\frac{\lambda}{\mu}\right)^n P_0}{n!} & \text{if } n < C \\[2ex] \dfrac{\left(\frac{\lambda}{\mu}\right)^n P_0}{C!C^{n-C}} & \text{if } n \geq C \end{cases}$$

Example:4.5.1

(1) A super market has 2 girls running up sales at the counters. If the service time for each customers is exponential with mean 4 minutes and if people arrive in Poisson fashion at the rate of 10 an hour.

(a) What is the probability of having to wait for service ?

(b) What is the expected percentage of idle time for each girl.

(c) If the customer has to wait, what is the expected length of his waiting time.

C = 2, λ = 1/6 per minute μ = ¼ per minute $\lambda / C\mu$ = 1/3.

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!} (\lambda/\mu)^n + \frac{1}{C!} (\lambda/\mu)^C \frac{C\mu}{C\mu - \lambda} \right]^{-1}$$

$$= \left[\sum_{n=0}^{C-1} \frac{1}{n!} (2/3)^n + \frac{1}{2!} (2/3)^2 \frac{2 x 1/4}{2 x 1/4 - 1/6} \right]^{-1}$$

$$= \left[\{1 + 2/3\} + 1/3 \right]^{-1} = 2^{-1} = 1/2$$

$\therefore$ P_0 = ½ & $P_1 = (\lambda/\mu) P_0$ = 1/3

a)

$$P[n \geq 2] = \sum_{n=2}^{\infty} P_n$$

$$= \frac{(\lambda/\mu)^C \mu_C}{C!(C\mu - \lambda)} P_0 \qquad \text{where } C = 2$$

$$= \frac{(2/3)^2 (1/2)}{2!(1/2 - 1/6)} (1/2) = 0.167$$

$\therefore$ Probability of having to wait for service = 0.167

b) Expand idle time for each girl = 1 - $\lambda/C\mu$

= 1 – 1/3 = 2/3 = 0.67 = 67%

Expected length of customers waiting time

$$= \frac{1}{C\mu - \lambda} = 3 \text{ minutes}$$

Example:4.5.2

There are 3 typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour, what fraction of time all the typists will be busy ? what is the average no of letters waiting to be typed ?

Here C = 3; λ = 15 per hour; μ = 6 per hour

P[all the 3 typists busy] = p[n ≥ m]

Where n = no. of customer in the system

$$P\left[n \geq C\right] = \frac{(\lambda/\mu)^C C\mu}{C![(\mu-\lambda)]} P_0$$

$$= \left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{1}{3!}(2.5)^3 \left[\frac{18}{18-15}\right]\right]^{-1}$$

= 0.0449

$$P\left[x \geq 3\right] \qquad = \frac{(2.5)^3(18)}{3!\,(18-15)}(0.0449)$$

= 0.7016

Average no of letters waiting to be typed

$$L_q \qquad = \left[\frac{1}{(C-1)!}\left(\frac{\lambda}{\mu}\right)^C \frac{\lambda\mu}{(C\mu-\lambda)^2}\right] P_0$$

$$= \left[\frac{(2.5)^3 90}{2!(18-15)^2}\right](0.0449)$$

= 3.5078

4.6 Model VI : (M/M/C) : (N / FCFS)

(Multiple Server, Limited Curved Model)

$$p_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^n \left(\frac{\lambda C}{\mu C - \lambda}\right)\right]^{-1}$$

4.6.1 Characteristics of the model

(1) Expected or average no. of customers in the queue

$$L_q = P_0\left(\lambda/\mu\right)^C \frac{P}{C!(1-P)^2} \qquad \left[1-\rho^{N-C}-(1-\rho)(N-C)\rho^{N-C}\right]$$

Where $\rho = \lambda / C\mu$

(2) Expected no of customers in the system

$$L_s = L_q + C - P_0 \sum_{n=0}^{C-1} \frac{C-n}{n!}\left(\lambda/\mu\right)^n$$

$$L_s = L_q + \frac{\lambda}{\mu}$$

Where $\lambda' = \mu\left[C \sum_{n=0}^{C-1}\left(C-n\right)P_n\right]$

(3) Expected waiting time in the system

$$W_q = W_s - \frac{1}{\mu} \quad \text{(or)}$$

$$W_q = \frac{L_q}{\lambda'} \qquad \text{Where } \lambda' = \mu\left[C - \sum_{n=0}^{C-1}(C-n)P_n\right]$$

(4) Expected waiting time in the queue

$$W_q = W_s - \frac{1}{\mu} \quad \text{(or)}$$

$$W_q = \frac{L_q}{\lambda'} \qquad \text{Where } \lambda' = \mu\left[C - \sum_{n=0}^{C-1}(C-n)P_n\right]$$

Example:4.6.1 A car service station has two bags where service can be offered simultaneously. Due to space limitation, only four cars are accepted for servicing. The arrived pattern is poission with a mean of one car every minute during the peak hours. The service time is exponential with mean 6 minutes. Find the average no of cars in eh system during peak hours, the average waiting time of a car and the average no of cars per hour that cannot enter the station because of full capacity.

λ = 1 car per minutes

μ = 1/6 per minute

C = 3, N = 7, $\rho = \lambda / C\mu = 2$

$$P_0 = \left[\sum_{n=0}^{3-1} \frac{6^n}{n!} + \sum_{n=3}^{7} \frac{1x6^n}{3^{n-3}(3!)}\right]^{-1} = \frac{1}{1141}$$

(i) Expected no of customers in the queue

$$L_q = \frac{(C\rho)^C \rho P_0}{C!(1-\rho)^2}\left[1-\rho^{N-C+1}-(1-\rho)(N-C+1)\rho^{N-C}\right]$$

$$= \frac{6^3 x\, 2}{3!(-1)^2}\left(\frac{1}{1141}\right)\left[1-2^5+5(2)^4\right]$$

$= 3.09$ Cars

(ii) Expected cars in the system

$$L_s = 3.09+3-P_0\sum_{n=0}^{2}\frac{(2-n)}{n!}(6^n)$$

$$= 0.06 \text{ Cars}$$

(iii) Expected waiting time of a car in the system

$$W_s = \frac{6.06}{1(1-p_7)} = \frac{0.66}{1-\frac{67}{3!3^4} x \frac{1}{1141}} = 12.3 \text{ min utes}$$

Since

$$P_n = \frac{1}{C!C^{n-c}}(\lambda / \mu)^{\lambda} P_0, \; C \le n \le N$$

(iv) Expected no of cars per hour at that cannot enter the station.

$$60\lambda P_N = 60, P_7 = \frac{6067}{3!3^4}\left(\frac{1}{1141}\right)$$

= 30.4 cars per hour

Example:4.6.2 A barber shop has two barbers and three chairs for customers. Assume that customers arrive in a poisson fashion at a rate of 5 per hour and that each barber services customers according to an exponential distribution with mean of 15 minutes. Further if a customer arrives and there are no empty chairs in the shop, he will leave. What is the probability that the shop is empty ? What is the expected no of customers in the shop ?

Here C = 2, N = 3,

λ = 5/60 = 1/12 customer / minute, μ = 1/15 customers / minute

$$P_0 = \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=2}^{3} \frac{1}{2^{n-2}(2!)}\left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$= \left[1 + \frac{1}{1!}\left(\frac{5}{4}\right) + \frac{1}{2!}\left(\frac{5}{4}\right)^2 + \frac{1}{4}\left(\frac{5}{4}\right)^3 \right]^{-1}$$

$$= \left[1 + \frac{5}{4} + \frac{25}{32} + \frac{125}{256} \right]^{-1} = 0.28$$

Probability that the shop is empty = probability that there are no customers in the system

P_0 = 0.28

Probability that there are n units in the system

$$P_n = \begin{cases} \frac{1}{n}\left(\frac{\lambda}{\mu}\right)^n P_0; & 0 \le n \le C \\ \frac{1}{C!C^{n-C}}\left(\frac{\lambda}{\mu}\right)^n P_0; & C \le n \le N \end{cases}$$

$$\therefore\ P_n = \begin{cases} \frac{1}{n!}\left(\frac{5}{4}\right)^n (0.28); 0 \le n \le 2 \\ \frac{1}{2!2^{n-2}}\left(\frac{5}{4}\right)^n (0.28); & 2 \le n \le 3 \end{cases}$$

The expected no of customers in the shop

$$L_S = L_q + C - P_0 \sum_{n=0}^{C-1} \frac{(C-n)\left(\frac{\lambda}{\mu}\right)^n}{n!}$$

$$= \sum_{n=2}^{3} (n-2)P_n + 2 - P_0 \sum_{n=0}^{2-1} \frac{(2-n)}{n!}\left(\frac{5}{4}\right)^n$$

$$= P_3 + 2 - (3.25)P_0$$

$$= \frac{(1.25)^3 (0.28)}{4} + 2 - (3.25)(0.28)$$

$$= 1.227 \text{ Customers (approx)}$$

4.7 Finite Source Models

Single-channel finite population model with Poisson arrivals and exponential service (M/M/I)(FCFS/n/M).

Characteristics of Finite Source Model (M/M/I) : FCFS/n/M

(1) Probability that the system is idle

$$P_0 = \left[\sum_{n=0}^{M} \frac{M!}{(M-n)!}\left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

(2) Probability that there are n customers in the system

$$P_0 = \frac{M!}{(M-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0, \ n = 0, 1, 2, ..., M$$

(3) Expected no of customers in the queue (or queue length)

$$L_q = M - \left(\frac{\lambda + \mu}{\lambda}\right)(1 - P_0)$$

(4) Expected no of customers in the system

$$L_q = M - \frac{\mu}{\lambda}(1 - P_0)$$

(5) Expected waiting time of a customer in the queue

$$W_q = \frac{L_q}{\mu(1-P_0)}$$

(6) Expected waiting time of a customer in the system

$$W_S = W_q + \frac{1}{\mu}$$

Example:4.7.1. A mechanic repairs machines. The mean time b/w service requirements is 5 hours for each machine and forms an exponential distribution. The mean repair time is C hour and also follows the same distribution pattern.

(i) Probability that the service facility will be idle

(ii) Probability of various no of machines (0 through 4) to be and being repaired

(iii) Expected no of machines waiting to be repaired and being repaired

(iv) Expected time a machine will wait in queue to be repaired.

g_n C = 1 (only one mechanic), λ = 1/5 = 0.2 Machine / hours

μ = 1 Machine / hour, μ = 4 Machines

$\rho = \lambda/\mu = 0.2$

(i) Probability that the system shall be idle (or empty) is

$$P_0 = \left[\sum_{n=0}^{M} \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu} \right)^n \right]^{-1}$$

$$= \left[\sum_{n=0}^{4} \frac{4!}{(4-n)!} (0.2)^n \right]^{-1}$$

$$= \left[1 + \frac{4!}{3!}(0.2) + \frac{4!}{2!}(0.2)^2 + \frac{4!}{1!}(0.2)^3 + \frac{4!}{6!}(0.2)^4 \right]^{-1}$$

$$= [1 + 0.8 + 0.48 + 0.192 + 0.000384]^{-1}$$

$$= (2.481)^{-1} \quad = 0.4030$$

(ii) Probability that there shall be various no of machines (0 through 5) in the system is obtained by using the formula

$$P_n = \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu} \right)^n P_0, \quad n \le M$$

$$P_0 = 0.4030$$

$$P_1 = \frac{4!}{3!}(0.2)(0.4030) = 0.3224$$

$$P_2 = \frac{4!}{2!}\,(0.2)^2 P_0 = 0.1934$$

$$P_3 = \frac{4!}{2!}\,(0.2)^3 P_0 = 0.0765$$

$$P_4 = 4!(0.2)^4 \; P_0 = 0$$

(iii) The expected no of machines to be and being repaired (in the system)

$$L_s = M - \frac{\mu}{\lambda}(1 - P_0)$$

$$= 4 - \frac{1}{0.2}(1 - 0.403)$$

$$= 1.015 \min utes$$

(iv) Expected time the machine will wait in the queue to be repaired

$$W_q = \frac{1}{\mu}\left[\frac{\mu}{1 - P_0} - \frac{\lambda + \mu}{\lambda}\right]$$

$$= \frac{4}{0.597} - 6$$

$$= 0.70 \text{ hours (or) } 42 \min utes$$

Example:4.7.2. Automatic car wash facility operates with only one bay cars arrive according to a poisson is busy. If the service time follows normal distribution with mean 12 minutes and S.D 3 minutes, find the average no of cars waiting in the parking lot. Also find the mean waiting time of cars in the parking lot.

A = 1/15, E(T) = 12 min, V(T) = 9 min,

$$\mu = \frac{1}{E(T)} = \frac{1}{12}$$

By P-K formula

$$E(N_S) = L_s = E(T) + \frac{\lambda^2[V(T) + E^2(T)]}{2 - [1 - \lambda E(T)]}$$

$$= \frac{12}{15} + \frac{\frac{1}{225}(9 + 144)}{2(1 - \frac{12}{15})}$$

$$= \frac{4}{5} + \frac{153}{90} \qquad = 2.5 \text{ Cars}$$

By Little's formula

$$E(N_q) = L_q = L_s - \frac{\lambda}{\mu}$$

$$L_q = 2.5 - \frac{12}{15}$$

$$= 1.7 \text{ cars}$$

$\therefore$ The average no of cars waiting in the parking lot = 1.7 cars

The mean waiting time of the cars in the parking lot

$$W_q = \frac{L_q}{\lambda} = \frac{1.7}{1/15} = 25.5 \text{ min utes}$$

(or) 0.425 hour

TUTORIAL PROBLEMS

1.Thelocalone-personbarber shopcana ccommodate amaximumof 5peopleatAtime(4waitingand1gettinghair

Customers arrive accordingtoaPoissondistributionwithmean5 perhour. The barbercutshair atanaverage rateo

ervicetime).

(a) What percentage of time is the barber idle?

(b) Whatfractionof thepotentialcustomers areturned away? (c) What isthe expectednumber of customers wai

hair cut?

(d) Howmuchtime cana customer expecttospendinthe barber shop?

2. Abank has twotellers workingon savings accounts.The first

tellerhandles withdrawalsonly.The second teller handlesdeposits only.It hasbeenfoundthattheservice time of

customer.Depositorsare foundtoarrive ina Poissonfashion throughoutthedaywithmeanarrivalrateof 16 per ho

Withdrawers alsoarrive ina Poissonfashionwithmeanarrivalrate of 14 per hour.Whatwouldbe the effectont
customers if eachteller couldhandle bothwithdrawals anddeposits.Whatwould bethe effect,ifthis couldonlybe acco
time to3.5 min.?

3. Customers arrive ataone-man barber shopaccording toaPoisson process mean interarrivaltime of 12mi
10minin thebarber's chair.

(a) Whatis the expectednumberof customers inthe barber

shopandinthe queue?

(b)Calculatethe percentageof time anarrivalcanwalk straightintothe barber'schair withouthavingtowait.

©howmuch time cana customer expecttospendinthe barber's shop?

(e)Managementwillprovide another chairandhire anotherbarber, whena customer's waitingtime intheshop ex

1.25h.Howmuchmustthe average rateof arrivalsincrease to warranta second barber?

(f)Whatis theprobabilitythatthe waitingtime in thesystemis greaterthan30min?

4. A2-personbarbershophas 5chairstoaccommodate waiting customers.Potentialcustomers, whoarrive
withoutentering barber shop.Customers arriveatthe average rateof 4perhour andspendanaverageof12min
P_0,P_1,P_7E(N_q) andE(W).

5. Derive the differenceequations for a Poissonqueuesystemin the steadystate

6. There are3 typists inanoffice.Each typistcan type anaverageof6 lettersperhour.If letters arrive for being
hour.

1.Whatfractionof thetime all the typists will bebusy?

2.What isthe average number of letters waitingtobetyped?

3.What isthe average time a letter has tospendfor waitingandforbeing typed?

4.What isthe probabilitythata letter willtake longer than20 min. waitingtobe typedand being typed?

7. Determine the steadystateprobabilities for M/M/C queueing system.

WORKED OUT EXAMPLE

Example 1: A super market has 2 girls running up sales at the counters. If the service time for each custor minutes and if people arrive in Poisson fashion at the rate of 10 an hour.

(a) What is the probability of having to wait for service ?

(b) What is the expected percentage of idle time for each girl.

(c) If the customer has to wait, what is the expected length of his waiting time.

C = 2, λ = 1/6 per minute μ = ¼ per minute $\lambda / C\mu$ = 1/3.

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!} (\lambda/\mu)^n + \frac{1}{C!} (\lambda/\mu)^C \frac{C\mu}{C\mu - \lambda} \right]^{-1}$$

$$= \left[\sum_{n=0}^{C-1} \frac{1}{n!} (2/3)^n + \frac{1}{2!} (2/3)^2 \frac{2 x 1/4}{2 x 1/4 - 1/6} \right]^{-1}$$

$$= \left[\{1 + 2/3\} + 1/3 \right]^{-1} = 2^{-1} = 1/2$$

$\therefore$ P_0 = ½ & $P_1 = (\lambda/\mu) P_0$ = 1/3

a)

$$P[n \geq 2] \quad = \sum_{n=2}^{\infty} P_n$$

$$= \frac{(\lambda/\mu)^C \mu_C}{C!(C\mu - \lambda)} P_0 \qquad \text{where } C = 2$$

$$= \frac{(2/3)^2 (1/2)}{2!(1/2 - 1/6)} (1/2) = 0.167$$

$\therefore$ Probability of having to wait for service = 0.167

b) Expand idle time for each girl = 1 - $\lambda/C\mu$

= 1 – 1/3 = 2/3 = 0.67 = 67%

Expected length of customers waiting time

$$= \frac{1}{C\mu - \lambda} = 3 \text{ minutes}$$

Example 2: There are 3 typists in an office. Each typist can type an average of 6 letters per hour. If letters
of 15 letters per hour, what fraction of time all the typists will be busy ? what is the average no of letters waiting to

Here C = 3; λ = 15 per hour; μ = 6 per hour

P[all the 3 typists busy] = p[n $\geq$ m]

Where n = no. of customer in the system

$$P\left[n \geq C\right] = \frac{(\lambda/\mu)^C C\mu}{C![(\mu-\lambda)]} P_0$$

$$= \left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{1}{3!}(2.5)^3\left[\frac{18}{18-15}\right]\right]^{-1}$$

= 0.0449

$$P\left[x \geq 3\right] \qquad = \frac{(2.5)^3(18)}{3!\,(18-15)}(0.0449)$$

= 0.7016

Average no of letters waiting to be typed

$$L_q \qquad = \left[\frac{1}{(C-1)!}\left(\frac{\lambda}{\mu}\right)^C \frac{\lambda\mu}{(C\mu-\lambda)^2}\right] P_0$$

$$= \left[\frac{(2.5)^3 90}{2!(18-15)^2}\right](0.0449)$$

= 3.5078

UNIT V

ADVANCED QUEUEING MODELS

5.1 Non-Markovian queues and Queue Networking The M/G/1 queueing system (M/G/1) : (∞/GD) mdel Pollaczek Ishintchine Formula

Let N and N1 be the numbers of customers in the system at time t and t+T, when two consecutive customers have just left the system after getting service.

Let k be the no. of customer arriving in the system during the service time T.

$$N^1 = \begin{cases} K & \text{if } n = 0 \\ (N-1)+11 & \text{if } n > 0 \end{cases}$$

Where k = 0, 1, 2, 3, … is the no. of arrivals during the service time (K is a discrete random variable)

Alternatively, if $\delta = \begin{cases} 1 & \text{if } N = 0 \\ 0 & \text{if } N < 0 \end{cases}$

Then $N^1 = N-1+\delta + k$.

Various formula for (M/G/1) : (∞/GD)
Model can be summarized as follows:

1) Average no. of customer in the system

$$L_S = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho \qquad \text{where } \sigma^2 = V(T),\ P = \lambda E(T) \text{ (or) } \rho = \lambda/4$$

2) Average queue length

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

3) Average waiting time of a customer in the queue

$$W_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)}$$

4) Average waiting time of a customer spends in the system

$$W_S = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} + \frac{1}{\mu}$$

Example :5.1.1

Automatic car wash facility operates with only one bay cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking let if the bay is busy. The parking lot is large enough to accommodate any no. of cars. If the service time for all cars is constant and equal to community determine.

(1) Mean no. of customers in the system L_S

(2) Mean no. of customers in the queue L_q

(3) Mean waiting time of a customer in the system W_S

(4) Mean waiting time of a customer in the system W_q

This is (M/G/I) : (∞/GD) model. Hence λ = 4 cars / hour.

T is the service time & is constant equal to 10 minutes

Then E(T) = 10 minutes & V(T) = 0.

$\therefore \frac{1}{\mu} = 10 \Rightarrow \mu = \frac{1}{10}$ per minute

$\therefore$ μ = 6 cars / hours and σ^2 = Var (T) = 0

$\rho = \lambda/\mu = 4/6 = 2/3$

Avg. no. of customers in the system

$$L_S = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho = 1.333 \approx 1 \text{ car}.$$

Avg. No. of customers in the queue.

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} = 0.667 \text{ cars.}$$

Avg. waiting time of a customer in the system

$$W_S = \frac{L_S}{\lambda} = 0.333 \text{ hour}$$

Avg. waiting time of a customer in the queue

$$W_q = \frac{L_q}{\lambda} = 0.167 \text{ hour}$$

Example :5.1.2 A car wash facility operates with only one day. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the factory's parking lot in the bay is busy. The parking lot is large enough to accommodate any no. of cars. If the service time for a car has uniform distribution b/w 8 & 12 minutes, find (i) The avg. no. of cars waiting in the parking lot and (ii) The avg. waiting time of car in the parking lot.

Solution

$\lambda = 4; \quad \lambda = 1/15$ cars / minutes

E(T) = Mean of the uniform distribution in (8, 12)

$= \frac{8+12}{2} = 10$ minutes $\qquad \left[\text{Mean} = \frac{a+b}{2}\right]$

V(T) $= 1/2\,(b-a)^2 = 4/3$

$\Rightarrow \mu = 1/10$ cars / minutes and σ^2 = 4/3

Then $\rho = \lambda/\mu = 2/3$

By ρ – k formula

$$L_q = \frac{\lambda^2\sigma^2+\rho^2}{\lambda(1-\rho)} = \frac{\frac{1}{225}\left(\frac{4}{3}\right)+\frac{4}{9}}{2\left(1-\frac{2}{3}\right)}$$

$$= \frac{\frac{4}{3}\left[\frac{1}{225}+\frac{1}{3}\right]}{\frac{2}{3}}$$

= 0.675 cars

$\therefore$ The avg. no. of cars waiting in the parking lot = 0.675 crs

The avg. waiting time of a car in the parking lot

$$= \frac{L_q}{\lambda} = 0.675 \text{ x } 15$$

W_q = 10.125 minutes .

5.2 QUEUE NETWORKS

Queue in series without possibility of queueing steady - state probability

P(0, 0) = Prob. (that both stages are empty

P(1, 0) = Prob. (that the 1st stage is full and the second is empty)

P(1,1) = Prob. (that both the stages are full, first is working)

P(b_1, 1) = Prob. (that first is blocked and the second is full)

P(0, 1) = Prob. (that 1st stage is empty and the second is full)

5.2.1 STEADY STATE EQUATION

$\lambda P(0,0) = \mu_2 P(0,1)$

$(\lambda+\mu_2)P(0,1) = \mu_1 P(0,1) + \mu_2 P(b,1)$

$\mu_1 P(1,0) = \mu_2 P(1,1) + \lambda P(0,0)$

$(\mu_1+\mu_2)P(1,1) = \lambda P(0,1)$

$\mu_2 P(b,1) = \mu_1 P(1,1)$

$P(0,0)+P(0,1)+P(1,0)+P(1,1)+P(b,1) = 1$

The solution of these equation is given by

P(0, 0) = $\lambda^2\mu_1(\mu_1+\mu_2)/\sigma$

P(0, 1) = $\lambda\mu_1\mu_2(\mu_1+\mu_2)/\sigma$

P(1, 0) = $\lambda\mu_2^2(\lambda+\mu_1+\mu_2)/\sigma$

P(1, 1) = $\lambda^2\mu_1\mu_2/\sigma$

P(b, 1) = $\lambda^2\mu_1^2/\sigma$

Where $\sigma = \mu_1(\mu_1+\mu_2)(\lambda^2+\lambda\mu_2+\mu_2^2)+\lambda(\mu_1+\mu_2+\lambda)\mu_2^2$

Hence the rate of loss - call is given by

L = P(1, 0) + P(1, 1) + P(0, 1)

Example :5.2.1 A repair facility shared by a large no. of machines has two sequential stations with respective rates one per hour and two per hour. The cumulative failure rate of all machines is 0.5 per hour. Assuming that the system behaviour may be approximated by the two-stage tandem queue, determine (i) the avg repair time (ii) the prob. that both service stations are idle (iii) the station which is the bottleneck of the service facility.

Given

$\lambda = 0.5 \qquad \mu_0 = 1 \qquad \mu_1 = 2$

$$\rho_0 = \lambda/\mu_0 = 0.5 \qquad \& \qquad \rho_1 = \lambda/\mu_1 = 0.25$$

The average length of the queue at station i(i = 0.1) is given by

$$E(N_i) = \frac{\rho_i}{1-\rho_i}$$

$$\therefore\ E(N_0) = 1 \qquad \& \qquad E(N_1) = 1/3$$

Using Little's formula, the repair delay at the two stations is respectively given by

$$E(R_0) = \frac{E(N_1)}{\lambda} = 2 \qquad \& \qquad E(R_1) = \frac{E(N_1)}{\lambda} = 2/3 \text{ hours}$$

Hence the avg. repair time is given by

$$E(R) = E(R_0) + E(R_1)$$

$$= 2 + 2/3 = 8/3 \text{ hours}$$

This can be decomposed into waiting time at station 0(=1 hour), the service time at station $0\left(= 1/N_0 = 1\right)$, the waiting time at station $1\left(= 1/6 \text{ hour}\right)$ and the service time at station $1\left(1/\mu_1 = 1/2 \text{ hour}\right)$.

The prob. that both service stations are idle = P(0, 0) = $(1 - \rho_0)(1 - \rho_1)$

$$= 3/8$$

Station 0 is the bottleneck of the repair facility since $P_0 = 0.5$ is the largest value.

Open Central Service Queueing Model

$$E(N_j) = \frac{\rho_j}{1-\rho_j} \qquad \text{and} \qquad E(R_j) = \frac{1}{\lambda}\frac{\rho j}{1-\rho_j}$$

Example :5.2.2

Consider the open central server queueing model with two I/O channels with a common service rate of 1.2 sec^{-1}. The CPU service rate is 2 sec^{-1}, the arrival rate is $1/7$ jobs / second. The branching prob. are given by $P_0 = 0.7$, $P_1 = 0.3$ and $P_2 = 0.6$. Determine the steady state prob., assuming the service times are independent exponentially distributed random variables.

Determine the queue length distributions at each node as well as the avg. response time from the source on the sink.

We are given

$\mu_1 = 1.2$ / second

$\mu_2 = 1.2$ / second

$\mu_0 = 2$ / second & $\lambda = 1/7$ jobs / second

The branching prob. are $P_0 = 0.7$, $P_1 = 0.3$ and $P_2 = 0.6$.

$$\lambda_0 = \frac{\lambda}{P_0}$$

$$\lambda_0 = \frac{1/7}{0.7}$$

$$= \frac{1}{4}9 = 10/49$$

$$\lambda_1 = \frac{\lambda P_1}{P_0} = 1/7 \text{ x } 0.3/0.7$$

$$= 3/49$$

$$\lambda_2 = \frac{\lambda P_2}{P_0} = 1/7 \text{ x } 0.6/0.7$$

$$= 6/49$$

The utilization ρ_j of node j is given by

$$\rho_j = \frac{\lambda_i}{\mu_j} \quad (j = 0,1,2)$$

$$\rho_0 = \frac{\lambda_0}{\mu_0} = 5/49$$

$$\rho_1 = \lambda_1/\mu_1 = 5/98$$

$$\rho_2 = \lambda_2/\mu_2 = 5/49$$

The steady state prob. are given by

$$P_j(k_j) = (1-\rho_j)\rho_j^{k_j} \text{ at node j}$$

$$P_0(k_0) = \frac{44}{49}\left(\frac{5}{49}\right)k_0$$

$$P_1(k_1) = (1-\rho_1)\rho_1^{k_1}$$

$$= \frac{93}{98}\left(\frac{5}{98}\right)^{k_1}$$

$$P_2(k_2) = (1-\rho_2)\rho_2^{k_2}$$
$$= \frac{44}{49}\left(\frac{5}{49}\right)^{k_2}$$

The average queue length $E(N_j)$ of node j is given by

$$E(N_j) = \frac{\rho_j}{1-\rho_j}$$

$\therefore$ For node 0,

$$E(N_0) = \frac{\rho_0}{1-\rho_0} = 5/44 \text{ job / second.}$$

For node 1,

$$E(N_1) = \frac{\rho_1}{1-\rho_1} = 5/93 \text{ job / second.}$$

For node 2,

$$E(N_2) = \frac{\rho_2}{1-\rho_2} = 5/44 \text{ job / second.}$$

The average response time from the source to the sink is given by

$$E(R) = \frac{1}{\mu_0 P_0 - \lambda} + \sum_{j=1}^{m} \frac{1}{\frac{\mu_0 P_0}{\rho_j} - \lambda}$$

$$= \frac{1}{1.4 - 1/7} + \frac{1}{7/3(1.2) - 1/7} + \frac{1}{7/6(1.2) - 1/-7}$$

$$= \frac{7}{8.8} + \frac{21}{55.8} + \frac{21}{26.4}$$

= 0.7954 + 0.3763 + 0.7954

= 1.9671

$\square$ 1.97 seconds.

TUTORIAL PROBLEMS

1. Derive the Balance equation of the birth and death process.

2. Derive the Pollaczek-Khinchine formula.

3. Consider a single server, poisson input queue with mean arrival rate of 10hour currently the server works according to an exponential distribution with

mean service time of 5minutes. Management has a training course which will result in an improvement in the variance of the service time but at a slight increase in the mean. After completion of the course;, its estimated that the mean service time willincreaseto5.5minutes but the standard deviation will decrease from5minutes to4minutes.Managementwouldliketoknow;whethertheyshould have the server undergo further training.

4. Ina heavy machine shop, the over head crane is 75%utilized.Time study observations gave the average slinging time as 10.5 minutes with a standard deviation of8.8minutes.What is the average call ingrate for the services of the crane and what is the average delay in getting service? If the average service time is cut to8.0minutes, with standard deviationof6.0minutes,howmuch reduction will occur, on average, in the delay of getting served?

5. Automatic car wash facility operates with only on Bay. Cars arrive according toa Poisson process, with mean of 4carsperhour and may wait in the facility's parking lot if the bay is busy. If the service time for all cars is constant and equal to 10min,determine L_S, L_q,W_S and W_q

WORKED OUT EXAMPLES

Example :1 A car wash facility operates with only one day. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the factory's parking lot in the bay is busy. The parking lot is large enough to accommodate any no. of cars. If the service time for a car has uniform distribution b/w 8 & 12 minutes, find (i) The avg. no. of cars waiting in the parking lot and (ii) The avg. waiting time of car in the parking lot.

Solution

$$\lambda = 4; \quad \lambda = \tfrac{1}{15} \text{ cars / minutes}$$

E(T) = Mean of the uniform distribution in (8, 12)

$$= \frac{8+12}{2} = 10 \text{ minutes} \qquad \left[\text{Mean } = \frac{a+b}{2}\right]$$

$$V(T) \;=\; \tfrac{1}{2}(b-a)^2 = \tfrac{4}{3}$$

$$\Rightarrow \mu \;=\; \tfrac{1}{10} \text{ cars / minutes and } \sigma^2 = 4/3$$

Then $\rho = \lambda/\mu = 2/3$

By ρ – k formula

$$L_q = \frac{\lambda^2\sigma^2+\rho^2}{\lambda(1-\rho)} = \frac{\tfrac{1}{225}\left(\tfrac{4}{3}\right)+\tfrac{4}{9}}{2\left(1-\tfrac{2}{3}\right)}$$

$$= \frac{\frac{4}{3}\left[\frac{1}{225} + \frac{1}{3}\right]}{\frac{2}{3}}$$

$= 0.675$ cars

$\therefore$ The avg. no. of cars waiting in the parking lot = 0.675 crs

The avg. waiting time of a car in the parking lot

$$= \frac{L_q}{\lambda} = 0.675 \text{ x } 15$$

$W_q \quad = 10.125$ minutes .

2-MARKS
Unit I Random Variable

1.. Define Random Variable (RV).

A random variable is a function X: S$\rightarrow$R that assigns a real number X(S) to every element $s \in S$, where S is the sample space corresponding to a random experiment E.

Ex: Consider an experiment of tossing an unbiased coin twice. The outcomes of the experiment are HH, HT, TH,TT.let X denote the number of heads turning up. Then X has the values 2,1,1,0. Here X is a random variable which assigns a real number to every outcome of a random experiment.

2. Define Discrete Random Variable.

If X is a random variable which can take a finite number or countably infinite number of pvalues, X is called a discrete RV.

Ex. Let X represent the sum of the numbers on the 2 dice, when two dice are trown.

3. Define Continuous Random Variable.

If X is a random variable which can take all values (i.e., infinite number of values) in an interval, then X is called a continuous RV.

Ex. The time taken by a lady who speaks over a telephone.

4. Define One-dimensional Random Variables.

If a random variable X takes on single value corresponding to each outcome of the experiment, then the random variable is called one-dimensional random variables.it is also called as scalar valued RVs.

Ex:

In coin tossing experiment, if we assume the random variable to be appearance of tail, then the sample space is{H,T} and the random variable is{1,0}.which is an one-dimensional random variables.

5. State the Properties of expectation.

If X and Y are random variables and a,b are constants, then

1.E(a)=a

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

$$E(a)=\sum_{i=1}^{n} a p_i = a\sum_{i=1}^{n} p_i = a(1)\ \left(\because \sum_{i=1}^{n} p_i = 1\right)$$

$$E(a) = a$$

2.E(aX)=aE(X)

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

$$E(aX)=\sum_{i=1}^{n} a x_i p_i = a\sum_{i=1}^{n} x_i p_i = aE(X)$$

3.E(aX+b)=aE(X)+b

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

$$E(aX+b)=\sum_{i=1}^{n}(ax_i+b)p_i=\sum_{i=1}^{n}(ax_i)p_i+\sum_{i=1}^{n}bp_i=a\sum_{i=1}^{n}x_ip_i+b\sum_{i=1}^{n}p_i$$

$$E(aX+b)=a\,E(X)+b \quad \{\because \sum_{i=1}^{n}p_i=1\}$$

4.E(X+Y)=E(X)+E(Y)

5. E(XY)=E(X).E(Y), if X and Y are random variables.

6. $E(X-\overline{X})=E(X)-\overline{X}=\overline{X}-\overline{X}=0$

6. A RV X has the following probability function

Values of X	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
P(x)	**a**	**3a**	**5a**	**7a**	**9a**	**11a**	**13a**	**15a**	**17a**

1) Determine the value of a.

2) Find P(X<3), P(X$\geq$3), P(0<X<5).

Solution:

1) We know that $\sum_x P(x)=1$

a+3a+5a+7a+9a+11a+13a+15a+17a=1

81a = 1

a = 1/81

2) P(X<3) = P(X=0) + P(X=1) +P(X=2)

= a+3a+5a

=9a =9/81=1/9

P(X$\geq$3) = 1- P(X<3) =1-1/9 = 8/9

P(0<X<5) = P(X=1) + P(X=2) + P(X=3) +P(X=4)

= 3a+5a+7a +9a =24a= 24/81

7. If X is a continuous RV whose PDF is given by

$$f(x)=\begin{cases} c(4x-2x^2), 0<x<2 \\ 0, \qquad otherwise \end{cases}$$

Find c.and mean of X.

Solution:

We know that $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_0^2 c(4x-2x^2)dx=1$$

c = 3/8

$$E(X)=\int_{-\infty}^{\infty} xf(x)dx=\int_0^2 \frac{3}{8}x(4x-2x^2)dx=\frac{8}{3}$$

8. A continuous RV X that can assume any value between x = 2 and x = 5 has a density function given by $f(x)=k(1+x)$. Fnd *P(X<4).*

Solution:

We know that $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_2^5 k(1+x)dx=1$$

k = 2/27

$$P(X<4)=\int_2^4 \frac{2}{27}(1+x)dx=\frac{16}{27}$$

9. A RV X has the density function

$$f(x) = \begin{cases} k\dfrac{1}{1+x^2}, -\infty < x < \infty \\ 0, \qquad otherwise \end{cases}$$ **. Find k .**

Solution:

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} k\frac{1}{1+x^2}dx = 1$$

$$k(\tan^{-1} x)_{-\infty}^{\infty} = 1$$

$$k\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1$$

$$\therefore k = \frac{1}{\pi}$$

10. If the p.d.f of a RV .X is given by $f(x) = \begin{cases} \frac{1}{4}, -2 < X < 2 \\ 0, elsewhere \end{cases}$ **.Find** $P[|X|>1]$.

Answer:

$$P[|X|>1] = 1 - P[|X|<1] = 1 - \int_{-1}^{1}\frac{1}{4}dx = 1 - \frac{1}{4}[1+1] = 1 - \frac{1}{2} = \frac{1}{2}$$

11. If the pdf of a RV X is $f(x) = \frac{x}{2}$ **in** $0 \le x \le 2$,find $P[X > 1.5 / X > 1]$

Answer:

$$P[X > 1.5 / X > 1] = \frac{p[X>1.5]}{P[X>1]} = \frac{\int_{1.5}^{2}\frac{x}{2}dx}{\int_{1}^{2}\frac{x}{2}dx} = \frac{4-2.25}{4-1} = 0.5833$$

12. Determine the Binomial distribution whose mean is 9 and whose SD is 3/2

Ans : np = 9 and npq = 9/4 $\therefore$ q = $\frac{npq}{np} = \frac{1}{4}$

$\Rightarrow$ p = 1- q = $\frac{3}{4}$. np = 9 $\Rightarrow$ n = $9\left(\frac{4}{3}\right)$=12.

P [x = r] = $12C_r\left(\frac{3}{4}\right)^r\left(\frac{1}{4}\right)^{12-r}$, r = 0,1,2,…,12.

13. Find the M.G.F of a Binomial distribution

$$M_x(t) = \sum_{i=0}^{n} e^{tx}\,{}_nC_x p^x q^{n-x} = \sum_{x=0}^{n} {}_nC_r (pe^t)^x q^{n-x} = (q+pe^t)^n$$

14. The mean and variance of the Binomial distribution are 4 and 3 respectively. Find P(X=0).

Ans :

mean = np = 4, Variance = npq = 3

$q = \frac{3}{4}$, $p = 1 - \frac{3}{4} = \frac{1}{4}$, np = 4 $\Rightarrow$ n = 16

P (X=0) $= {}_nC_0 p^0 q^{n-0} = 16C_0 p^0 q^{16-0} = \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{16} = \left(\frac{3}{4}\right)^{16}$

15. For a Binomial distribution mean is 6 and standard deviation is $\sqrt{2}$. Find the first two terms of the distribution.

Ans : Given np = 6, npq $= \left(\sqrt{2}\right)^2 = 2$

$q = \frac{2}{6} = \frac{1}{3}$, $p = 1 - q = \frac{2}{3}$, np = 6 $\Rightarrow n\left(\frac{2}{3}\right) = 6 \Rightarrow n = 9$

P (X= 0) = ${}_nC_0 p^0 q^{n-0} = 9C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{9-0} = \left(\frac{1}{3}\right)^9$

P (X=1) = ${}_nC_1 p^1 q^{n-1} = 9\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8 = 6\left(\frac{1}{3}\right)^8$

16. The mean and variance of a binomial variate are 4 and $\frac{4}{3}$ respectively, find $P[X \geq 1]$.

Ans : np = 4, npq $= \frac{4}{3} \Rightarrow q = \frac{1}{3}, p = \frac{2}{3}$

$P[X \geq 1]$= 1- P[X<1] = 1 – P[X = 0] =1-$\left(\frac{1}{3}\right)^6$ = 0.9986

17. For a R.V X, $M_x(t) = \frac{1}{81}(e^t + 2)^4$. Find $P(X \leq 2)$.

Sol: Given $M_x(t) = \frac{1}{81}(e^t + 2)^4 = \left(\frac{e^t}{3} + \frac{2}{3}\right)^4$ ------------------- (1)

For Binomial Distribution, $M_x(t) = (q + pe^t)^n$. ------------------------ (2)

Comparing (1)&(2),

$\therefore n = 4,\ q = \frac{2}{3},\ p = \frac{1}{3}.$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = 4C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 + 4C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 + 4C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)$$

$$= \frac{1}{81}(16 + 32 + 24) = \frac{72}{81} = 0.8889.$$

18. If a R.V X takes the values -1,0,1 with equal probability find the M.G.F of X.

Sol: P[X= -1]=1/3 , P[X=0]=1/3, P[X=1]=1/3

$$M_x(t) = \sum_x e^{tx} P(X = x) = \frac{1}{3}e^{-t} + \frac{1}{3} + \frac{1}{3}e^t = \frac{1}{3}(1 + e^t + e^{-t}).$$

19. A die is thrown 3 times. If getting a 6 is considered as success find the probability of atleast 2 success.

Sol: $p = \frac{1}{6}, q = \frac{5}{6} n = 3.$

P(at least 2 success)=P(X$\geq$2) = P(X=2)+P(X=3)

$$= 3C_2\left(\frac{1}{6}\right)^2 \frac{5}{6} + 3C_3\left(\frac{1}{6}\right)^3 = \frac{2}{27}.$$

20. Find p for a Binimial variate X if n=6,and 9P(X=4)=P(X=2).

Sol: $9P(X=4)=P(X=2) \Rightarrow 9\left({}_6C_4 p^4 q^2\right) = {}_6C_2 p^2 q^4$

$$\Rightarrow 9p^2 = q^2 = (1-p)^2 \;\therefore 8p^2+2p-1=0$$

$$\therefore p = \frac{1}{4}\left(\because p \neq -\frac{1}{2}\right)$$

21. Comment on the following

"The mean of a BD is 3 and variance is 4"

For B.D, Variance< mean

$\therefore$ The given statement is wrongs

22. Define poisson distribution

A discrete RV X is said to follow Poisson Distribution with parameter λ if its probability mass function is p(x) = $\dfrac{e^{-\lambda}\lambda^{x}}{x!}$, x = 0, 1,2, …… ∞

23. If X is a Poisson variate such that P(X=2)=9P (X = 4) + 90 P(X=6), find the variance

Ans : P [X = x] = $\dfrac{e^{-\lambda}\lambda^{x}}{x!}$

Given P(X=2)=9P (X = 4) + 90 P(X=6)

$$\therefore \frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\Rightarrow \frac{1}{2} = \frac{9}{24}\lambda^2 + \frac{90}{720}\lambda^4 \Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow (\lambda^2+4)(\lambda^2-1)=0$$

$$\Rightarrow \lambda^2 = -4 \quad \text{or } \lambda^2 = 1$$

hence $\lambda = 1[\because \lambda^2 \neq -4]$ Variance=1.

24. It is known that 5% of the books bound at a certain bindery have defective bindings. find the probability that 2 of 100 books bound by this bindery will have defective bindings.

Ans : Let X denote the number of defective bindings.

p = $\dfrac{5}{100}$ n = 100 $\qquad \therefore \lambda = np = 5$

P [X =2] = $\dfrac{e^{-\lambda}\lambda^2}{2!} = \dfrac{e^{-5}(25)}{2} = 0.084$

25. Find λ, if X follows Poisson Distribution such that P(X=2)=3P(X=3).

Sol: P(X=2)=3P(X=3) $\Rightarrow \dfrac{e^{-\lambda}\lambda^2}{2!} = \dfrac{3e^{-\lambda}\lambda^3}{3!} \Rightarrow \dfrac{1}{2} = \dfrac{3\lambda}{6} \Rightarrow \lambda = 1.$

26. If X is a Poisson variate such that $P(X=1)=\dfrac{3}{10}$ and $P(X=2)=\dfrac{1}{5}$.

Find $P(X=0)$ and $P(X=3)$.

Sol: $P(X=1)=\dfrac{3}{10} \quad \Rightarrow \dfrac{e^{-\lambda}\lambda}{1} = \dfrac{3}{10}$ ………..(1)

$P(X=2)=\dfrac{1}{5} \quad \Rightarrow \dfrac{e^{-\lambda}\lambda^2}{2} = \dfrac{1}{5}$ …………….(2)

$$\frac{(2)}{(1)} \Rightarrow \frac{\lambda}{2} = \frac{10}{15} \Rightarrow \lambda = \frac{4}{3} \qquad \therefore P(X=0) = \frac{e^{-\frac{4}{3}}\left(\frac{4}{3}\right)^0}{0!} = 0.2636$$

$$\therefore P(X=3) = \frac{e^{-\frac{4}{3}}\left(\frac{4}{3}\right)^3}{3!}$$

27. For a Poisson Variate X , $E(X^2) = 6$ .What is E(X).

Sol: $\lambda^2 + \lambda = 6 \Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = 2, -3.$

But $\lambda > 0 \therefore \lambda = 2$ Hence $E(X) = \lambda = 2$

28. A Certain Blood Group type can be find only in 0.05% of the people. If the population of a randomly selected group is 3000.What is the Probability that atleast a people in the group have this rare blood group.

Sol: p=0.05% =0.0005 n=3000 $\therefore \lambda = np = 1.5$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - e^{-1.5}\left[1 + \frac{1.5}{1}\right] = 0.4422.$$

29. If X is a poisson variate with mean λ show that E[X^2] = λ E[X+1]

E[X^2] = $\lambda^2 + \lambda$

E (X+1) = E[X] + 1 $\therefore E[X^2] = \lambda E[X+1]$

30. Find the M.G.F of Poisson Distribution.

Ans :

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty}\frac{(e^t\lambda)^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

31. A discrete RV X has M.G.F $M_x(t) = e^{2(e^t-1)}$. Find E (X), var (X) and P(X=0)

Ans : $M_X(t) = e^{2(e^t-1)} \Rightarrow$ X follows Poisson Distribution $\therefore \lambda = 2$

Mean = E (X) = $\lambda = 2$ Var (X) = $\lambda = 2$

$$P[X=0] = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-2}2^0}{0!} = e^{-2}$$

32. If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6th attempt?

Ans : Given p = 0.5 q = 0.5

By Geometric distribution

P [X =x] = q^x p, x = 0,1,2………

since the target is destroyed on 6th attempt x = 5

$\therefore$ Required probability = q^x p = $(0.5)^6$ = 0.0157

33. Find the M.G.F of Geometric distribution

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}q^x p = p\sum_{x=0}^{\infty}(qe^t)^x$$

$$= p[1-qe^t]^{-1} = \frac{p}{1-qe^t}$$

34. Find the mean and variance of the distribution P[X=x]=2^{-x}, x=1,2,3….

Solution:

$$P[X=x]= \frac{1}{2^x} = \left(\frac{1}{2}\right)^{x-1}\frac{1}{2}, x = 1,2,3.....$$

$$\therefore p = \frac{1}{2}\ and\ \ q = \frac{1}{2}$$

$$Mean = \frac{q}{p} = 1;\ Variance = \frac{q}{p^2} = 2$$

35. Find the expected value and the variance of the number of times one must throw a die until the outcome 1 has occurred 4 times.

Solution:

X follows the negative bionomial distribution with parameter r = 4 and p=1/6

E(X)= mean = r P = rqQ = r (1-p) (1/p)= 20. (p=1/Q and q=P/Q)

Variance = rPQ = r(1-p)/p^2 = 120.

36. If a boy is throwing stones at a target, what is the probability that his 10th throw is his 5th hit, if the probability of hitting the target at any trial is ½?

Solution:

Since 10th throw should result in the 5th successes ,the first 9 throws ought to have resulted in 4 successes and 5 faliures.

n = 5,r = 5, $p = \frac{1}{2} = q$

$\therefore$ Required probability = P(X=5)= (5+5-1)C_5 $(1/2)^5$ $(1/2)^5$

=9C_4 $(1/2^{10})$ = 0.123

37. Find the MGF of a uniform distribution in (a, b)?

Ans :

$$M_X(t) = \frac{1}{b-a}\int_a^b e^{tx}dx \qquad = \frac{e^{bt} - e^{at}}{(b-a)t}$$

38. Find the MGF of a RV X which is uniformly distributed over (-2, 3)

$$M_X(t) = \frac{1}{5}\int_{-2}^{3} e^{tx}dx = \frac{e^{3t} - e^{-2t}}{5t}\ for\ t \neq 0$$

39. The M.G.F of a R.V X is of the form M_x (t) = $(0.4e^t + 0.6)^8$ what is the M.G.F of the R.V, Y = 3X +2

$M_Y(t) = e^{2t}M_X(t) = e^{2t}\left((0.4)\ e^{3t} + 0.6\right)^8$

40. If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$ find P(X<0)

Ans : Let X follows uniform distribution in (a,b)

mean = $\frac{b+a}{2} = 1$ and Variance = $\frac{(b-a)^2}{12} = \frac{4}{3}$

$\therefore$ a+b = 2 $(b-a)^2 = 16 \Rightarrow b - a = \pm 4$

Solving we get a=-1 b=3

$\therefore$ f(X) = $\frac{1}{4}$, - 1 < x < 3

$\therefore$ P[X< 0] = $\int_{-1}^{0} f(x)dx = \frac{1}{4}$

41. A RV X has a uniform distribution over (-4, 4) compute $P(|X|> 2)$

Ans :

$$f(x) = \begin{cases} \frac{1}{8}, & -4 < x < 4 \\ 0 \text{ other wise} \end{cases}$$

$$P(|X|> 2) = 1 - P(|X| \le 2) = 1 - P(-2 < X < 2) = 1 - \int_{-2}^{2} f(x)dx = 1 - \frac{4}{8} = \frac{1}{2}$$

42. If X is Uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.Find the p.d.f of Y=tan X.

Sol: $f_X(x) = \frac{1}{\pi}$; $X = \tan^{-1} Y \Rightarrow \frac{dx}{dy} = \frac{1}{1+y^2}$.

$$\therefore f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \Rightarrow f_Y(y) = \frac{1}{\pi(1+y^2)}, \ -\infty < y < \infty$$

43. If X is uniformly distributed in (-1,1).Find the p.d.f of $y = \sin\frac{\pi x}{2}$.

Sol:

$$f_X(x) = \begin{cases} \frac{1}{2}, -1 < x < 1 \\ 0, otherwise \end{cases},$$

$$x = \frac{2\sin^{-1} y}{\pi} \Rightarrow \frac{dx}{dy} = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}} \quad \text{for} \quad -1 \le y \le 1$$

$$f_Y(y) = \frac{1}{2}\left[\frac{2}{\pi}\frac{1}{\sqrt{1-y^2}}\right] \Rightarrow f_Y(y) = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}} \quad \text{for} \quad -1 \le y \le 1$$

44.The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$ what is the probability that a repair takes at least 10 hours given that its duration exceeds 9 hours?

Ans :

Let X be the RV which represents the time to repair machine.

P[X≥ 10/ X ≥ 9] = P [X≥ 1] (by memory less property)

$$= \int_{1}^{\infty} \frac{1}{2} e^{-x/2} dx = 0.6065$$

45. The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{3}$ what is the probability that the repair time exceeds 3 hours?

Ans : X – represents the time to repair the machine

$$\therefore f(x) = \frac{1}{3}e^{-x/3} > 0$$

$$\text{P (X > 3)} = \int_3^\infty \frac{1}{3}e^{-x/3}dx = e^{-1} = 0.3679$$

46.Find the M.G.F of an exponential distribution with parameter λ.

Sol: $M_x(t) = \lambda\int_0^\infty e^{tx}e^{-\lambda x}dx = \lambda\int_0^\infty e^{-(\lambda-t)x}dx = \frac{\lambda}{\lambda-t}$

47. State the memory less property of the exponential distribution.

Soln: If X is exponential distributed with parameter λ then

$P(X > s+t / X > s) = P(X > t)$ for s , t >0

48. If X has a exponential distribution with parameter λ ,find the p.d.f of Y=log X.

Sol: Y=log X $\Rightarrow e^y = x \Rightarrow \frac{dx}{dy} = e^y$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|. \Rightarrow f_Y(y) = e^y\lambda e^{-\lambda e^y}.$$

49. If X has Exponential Distribution with parameter 1,find the p.d.f of $Y = \sqrt{X}$.

Sol: $Y = \sqrt{X} \Rightarrow X = Y^2 \quad f_X(x) = e^{-x}, x > 0.$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 2ye^{-x} = 2ye^{-y^2}, y > 0.$$

50 . Write the M.G.F of Gamma distribution

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}f(x)dx$$

$$= \frac{\lambda^\gamma}{\Gamma\gamma}\int_0^\infty e^{-(\lambda-t)x}x^{\gamma-1}dx \qquad = \qquad \frac{\lambda^\gamma}{\Gamma\gamma}\frac{\Gamma\gamma}{(\lambda-t)^\gamma}$$

$$\therefore Mx(t) = \left(1-\frac{t}{\lambda}\right)^{-\gamma}$$

51. Define Normal distribution

A normal distribution is a continuous distribution given by

$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$ where X is a continuous normal variate distributed with density function $f(X) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$ with mean μ and standard deviation σ

52. What are the properties of Normal distribution

(i) The normal curve is symmetrical when p = q or p $\approx$ q.

(ii) The normal curve is single peaked curve.

(iii) The normal curve is asymptotic to x-axis as y decreases rapidly when x increases numerically.

(iv) The mean, median and mode coincide and lower and upper quartile are equidistant from the median.

(v) The curve is completely specified by mean and standard deviation along with the value y_o.

53. Write any four properties of normal distribution.

Sol: (1) The curve is Bell shaped
(2) Mean, Median, Mode coincide
(3) All odd moments vanish
(4) x - axis is an asymptote of the normal curve

54. If X is a Normal variate with mean30 and SD 5.Find P [26<X<40].

Sol: P [26<X<40] =P $[-0.8 \le Z \le 2]$ where $Z = \dfrac{X-30}{5}$ $\left\{\because Z = \dfrac{X-\mu}{\sigma}\right\}$

$= P[0 \le Z \le 0.8] + P[0 \le Z \le 2]$
= 0.2881+0.4772
=0.7653.

55. If X is normally distributed RV with mean 12 and SD 4.Find P [X$\le$ 20].

Sol: $P[X \le 20] = P[Z \le 2]$ where $Z = \dfrac{X-12}{4}$ $\left\{\because Z = \dfrac{X-\mu}{\sigma}\right\}$

$= P[-\infty \le Z \le 0] + P[0 \le Z \le 2]$
=0.5+0.4772
=0.9772.

56. **If X is a N(2,3),find** $P\left[Y \ge \frac{3}{2}\right]$ **where Y+1=X.**

Answer:

$$P\left[Y \ge \frac{3}{2}\right] = P\left[X-1 \ge \frac{3}{2}\right] = P[X \ge 2.5] = P[Z \ge 0.17] = 0.5 - P[0 \le Z \le 0.17]$$
$$= 0.5 - 0.0675 = 0.4325$$

57.. If X is a RV with p.d.f $f(x) = \dfrac{x}{12}$ **in 1 < x < 5 and =0, otherwise.Find the p.d.f of Y=2X-3.**

Sol: Y=2X-3 $\Rightarrow \dfrac{dx}{dy} = \dfrac{1}{2}$

$f_Y(y) = f_X(x)\left|\dfrac{dx}{dy}\right| = \dfrac{y+3}{4}$,. in -1<y<7.

58.. If X is a Normal R.V with mean zero and variance σ^2 **Find the p.d.f of** $Y = e^X$ **.**

Sol: $Y = e^X \Rightarrow \log Y = X \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{y}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{y} f_X(\log y)$$
$$= \frac{1}{\sigma y\sqrt{2\pi}} \exp\left(-\left(\log y - \mu^2\right)/2\sigma^2\right)$$

59. If X has the p.d.f $f(x) = \begin{cases} x, 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$ **find the p.d.f of** $Y = 8X^3$.

Sol: $Y = 8X^3 \Rightarrow X = \frac{1}{2}Y^{\frac{1}{3}} \qquad \Rightarrow \frac{dx}{dy} = \frac{1}{6}y^{-\frac{2}{3}}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = (x)\left(\frac{1}{6}Y^{-\frac{2}{3}}\right) = \frac{1}{12}Y^{-\frac{1}{3}}, Y > 0$$

60. The p.d.f of a R.V X is $f(x) = 2x, 0 < x < 1$. **Find the p.d.f of** $Y = 3X + 1$.

Sol: $Y = 3X + 1 \Rightarrow X = \frac{Y-1}{3}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 2x\frac{1}{3} = 2\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right) = \frac{2(y-1)}{9}.,1 < y < 4.$$

Moment generating functions

1. Define n^{th} Moments about Origin

The n^{th} moment about origin of a RV X is defined as the expected value of the n^{th} power of X.

For discrete RV X, the n^{th} moment is defined as $E(X^n) = \sum_i x_i^n p_i = \mu_n', n \geq 1$

For continuous RV X, the n^{th} moment is defined as $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx = \mu_n', n \geq 1$

2.Define n^{th} Moments about Mean

The n^{th} central moment of a discrete RV X is its moment about its mean $\overline{X}$ and is defined as

$$E\left(X - \overline{X}\right)^n = \sum_i \left(x_i - \overline{X}\right)^n p_i = \mu_n, n \geq 1$$

The n^{th} central moment of a continuous RV X is defined as

$$E\left(X - \overline{X}\right)^n = \int_{-\infty}^{\infty} \left(x - \overline{X}\right)^n f(x)dx = \mu_n, n \geq 1$$

3..Define Variance

The second moment about the mean is called variance and is represented as σ_x^2

$$\sigma_x^2 = E\left[X^2\right] - \left[E(X)\right]^2 = \mu_2' - \left(\mu_1'\right)^2$$

The positive square root σ_x of the variance is called the standard deviation.

4.Define Moment Generating Functions (M.G.F)

Moment generating function of a RV X about the origin is defined as

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum_x e^{tx} P(x), ifXisdiscrete. \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx, ifXiscontinous. \end{cases}$$

Moment generating function of a RV X about the mean is defined as

$$M_{X-\mu}(t) = E(e^{t(x-\mu)})$$

5. Properties of MGF

1. $M_{X=a}(t) = e^{-at} M_X(t)$

Proof:

$$M_{X=a}(t) = E(e^{t(x-a)}) = E(e^{tx}.e^{-at}) = E(e^{tx})e^{-at} = e^{-at} M_X(t)$$

2. If X and Y are two independent RVs, then $M_{X+Y}(t) = M_X(t).M_Y(t)$

Proof:

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX+tY}) = E(e^{tX}.e^{tY}) = E(e^{tX}).E(e^{tY}) = M_X(t).M_Y(t)$$

3. $If M_X(t) = E(e^{tx}) then M_{cX}(t) = M_X(ct)$

Proof:

$$M_{cX}(t) = E(e^{tcX}) = E(e^{(ct)X}) = M_X(ct)$$

4. If Y=aX+b then $M_Y(t) = e^{bt} M_X(at)$ where $M_X(t)$=MGF of X.

5. If $M_{X_1}(t) = M_{X_2}(t)$ for all t, then $F_{X_1}(x) = F_{X_2}(x)$ for all x.

UNIT-I RANDOM VARIABLE

1. If the RV X takes the values 1, 2, 3 and 4 such that
$2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$, find the probability distribution and cumulative distribution function of X.

2. A RV X has the following probability distribution.

X:	-2	-1	0	1	2	3
P(x):	0.1	K	0.2	2K	0.3	3K

Find (1) K, (2) $P(X<2)$, $P(-2< X <2)$, (3) CDF of X, (4) Mean of X.

3. If X is RV with probability distribution

X:	1	2	3
P(X):	1/6	1/3	1/2

Find its mean and variance and $E(4X^3 +3X+11)$.

4. A RV X has the following probability distribution.

X:	0	1	2	3	4	5	6	7
P(x):	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find (1) K, (2) $P(X<2)$, $P(1.5 <X< 4.5/X>2)$, (3) The smallest value of λ for which $P(X \le \lambda) > 1/2$.

5. A RV X has the following probability distribution.

X:	0	1	2	3	4
P(x):	K	3K	5K	7K	9K

Find (1) K, (2) $P(X<3)$ and $P(0<X< 4)$, (3) Find the distribution function of X.

6. If the density function of a continuous RV X is given by $f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & Otherwise \end{cases}$

Find i)a ii)CDF of X.

7. A continuous RV X that can assume any value between x=2 and x=5 has a density function given by $f(x) = k(1+x)$. Fnd $P(X<4)$.

8. If the density function of a continuous RV X is given by $f(x) = kx^2 e^{-x}, x > 0$. Find k, mean and variance.

9. If the cdf of a continuous RV X is given by $F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3 - x^2), & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}$

Find the pdf of X and evaluate $P(|X| \le 1)$ and $P(\frac{1}{3} \le X < 4)$.

10. A continuous RV X has the pdf $f(x) = Kx^2 e^{-x}, x \ge 0$. Find the r^{th} moment about

origin.Hence find mean and variance of X.

11. Find the mean, variance and moment generating function of a binomialdistribution.

12. 6 dice are thrown 729 times.How many times do you expect atleast three dice to show 5or 6?

.13. It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the no. of packets containing atleast, exactly and atmost 2 defective items in a consignment of 1000 packets using (i) Binomial distribution

14. Find mean , variance and MGF of Geometric distribution.

15. The pdf of the length of the time that a person speaks over phone is

$$f(x) = \begin{cases} Be^{\frac{-x}{6}}, & x > 0 \\ 0, & otherwise \end{cases}$$

what is the probability that the person wil talk for (i) more than 8 minutes (ii)less than 4 minutes (iii) between 4 and 8 minutes.

16. State and prove the memory less property of the exponential distribution.

17. If the service life, in hours, of a semiconductor is a RV having a Weibull distribution with the parameters $\alpha = 0.025$ and $\beta = 0.5$,
    1. How long can such a semiconductor be expected to last?
    2. What is the probability that such a semiconductor will still be in operating condition after 4000 hours?

Unit II Two Dimensional Random Variables

1.Define Two-dimensional Random variables.

Let S be the sample space associated with a random experiment E. Let X=X(S) and Y=Y(S) be two functions each assigning a real number to each $s \in S$. Then (X,Y) is called a two dimensional random variable.

2. The following table gives the joint probability distribution of X and Y. Find the mariginal density functions of X and Y.

Y /X	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Answer:

The marginal density of X

$$P(X = x_i) = p_{i*} = \sum_j p_{ij}$$

X	1	2	3
P(X)	0.3	0.4	0.3

The marginal density of Y

$$P(Y = y_j) = p_{*j} = \sum_i p_{ij}$$

Y	1	2
P(Y)	0.4	0.6

3.If $f(x,y) = kxye^{-(x^2+y^2)}$, $x \ge 0, y \ge 0$ **is the joint pdf, find** k.

Answer:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1 \Rightarrow \int_0^{\infty}\int_0^{\infty} kxye^{-(x^2+y^2)}dydx = 1$$

$$k\int_0^{\infty} xe^{-x^2}dx\int_0^{\infty} ye^{-y^2}dx = 1 \Rightarrow \frac{k}{4} = 1$$

$$\therefore k = 4$$

4. Let the joint pdf of X and Y is given by $f(x,y) = \begin{cases} cx(1-x), 0 \le x \le y \le 1 \\ 0 \qquad\quad , otherwise \end{cases}$

Find the value of C.

Answer:

$$\int_0^1\int_0^y Cx(1-x)dxdy = 1 \Rightarrow \frac{C}{6}\int_0^1 (3y^2 - 2y^3)dy = 1 \Rightarrow \frac{C}{6}\left[1 - \frac{1}{2}\right] = 1$$

5. The joint p.m.f of (X,Y) is given by $P(x,y) = k(2x+3y),\ x = 0,1,2;\ y = 1,2,3.$ **.Find the marginal probability distribution of X.**

Answer:

X \Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

$$\sum_y\sum_x P(x,y) = 1 \Rightarrow 72k = 1 \therefore k = \frac{1}{72}$$

Marginal distribution of X:

X	0	1	2
P(X)	18/72	24/72	30/72

6. If X and Y are independent RVs with variances 8and 5.find the variance of 3X+4Y.

Answer:

Given Var(X)=8 and Var(Y)=5

To find:var(3X-4Y)

We know that Var(aX - bY)=a^2Var(X)+b^2Var(Y)

var(3X- 4Y)=3^2Var(X)+4^2Var(Y) =(9)(8)+(16)(5)=152

7. Find the value of k if $f(x,y) = k(1-x)(1-y)$ for $0 < x, y < 1$ is to be joint density function.

Answer:

We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$

$$\int_0^1\int_0^1 k(1-x)(1-y)dxdy = 1 \Rightarrow k\left[\int_0^1 (1-x)dx\right]\left[\int_0^1 (1-y)dy\right] = 1$$

$$k\left[x - \frac{x^2}{2}\right]_0^1\left[y - \frac{y^2}{2}\right]_0^1 = 1 \Rightarrow \frac{k}{4} = 1 \qquad \therefore k = 4$$

8. If X and Y are random variables having the joint p.d.f $f(x,y) = \frac{1}{8}(6-x-y),\ \ 0 < x < 2,\ 2 < y < 4$, find P(X<1,Y<3)

Answer:

$$P(X < 1, Y < 3) = \frac{1}{8}\int_0^1\int_2^3 (6-x-y)dydx = \frac{1}{8}\int_0^1\left(\frac{7}{2} - x\right)dx = \frac{3}{8}$$

9. The joint p.d.f of (X,Y)is given by $f(x,y) = \frac{1}{4}(1+xy), |x| < 1, |y| < 1$ and $= 0$,otherwise. Show that X and Y are not independent.

Answer:

Marginal p.d.f of X :

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-1}^{1}\frac{1}{4}(1+xy)dy = \frac{1}{2},\quad -1 < x < 1$$

$$f(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$$

Marginal p.d.f of Y :

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-1}^{1}\frac{1}{4}(1+xy)dx = \frac{1}{2},\quad -1 < y < 1$$

$$f(y) = \begin{cases} \frac{1}{2}, & -1 < y < 1 \\ 0, & otherwise \end{cases}$$

Since $f(x)f(y) \neq f(x,y)$,X and Y are not independent.

10.The conditional p.d.f of X and Y=y is given by $f\left(\frac{x}{y}\right) = \frac{x+y}{1+y}e^{-x}, 0 < x < \infty, 0 < y < \infty,$ find $P[X < 1/Y = 2]$.

Answer:

When y =2, $f(x/y=2) = \frac{x+2}{3}e^{-x}$

$$\therefore P[X < 1/Y = 2] = \int_0^1 \frac{x+2}{3}e^{-x}dx = \frac{1}{3}\int_0^1 xe^{-x}dx + \frac{2}{3}\int_0^1 e^{-x}dx = 1 - \frac{4}{3}e^{-1}$$

11. The joint p.d.f of two random variables X and Y is given by

$f(x,y)=\frac{1}{8}x(x-y), 0<x<2, -x<y<x$ **and** $=0$**, otherwise.**

Find $f(y/x)$

Answer:

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_{-x}^{x}\frac{1}{8}x(x-y)dy=\frac{x^3}{4},\quad 0<x<2$$

$$f(y/x)=\frac{f(x,y)}{f(x)}=\frac{x-y}{2x^2}, -x<y<x$$

12. If the joint pdf of (X,Y) is $f(x,y)=\frac{1}{4},\ 0\le x,\ y<2$ **,find** $P[X+Y\le 1]$

Answer:

$$P[X+Y\le 1]=\int_0^1\int_0^{1-y}\frac{1}{4}dxdy=\frac{1}{4}\int_0^1(1-y)dy=\frac{1}{8}.$$

13. If the joint pdf of (X,Y) is $f(x,y)=6e^{-2x-3y},\ x\ge 0, y\ge 0,$ **find the conditional density of Y given X.**

Answer:

Given $f(x,y)=6e^{-2x-3y},\ x\ge 0, y\ge 0,$

The Marginal p.d.f of X:

$$f(x)=\int_0^{\infty}6e^{-2x-3y}dy=2e^{-2x},\ x\ge 0$$

Conditional density of Y given X:

$$f(y/x)=\frac{f(x,y)}{f(x)}=\frac{6e^{-2x-3y}}{2e^{-2x}}=3e^{-3y}, y\ge 0.$$

14.**Find the probability distribution of (X+Y) given the bivariate distribution of (X,Y).**

X \Y	**1**	**2**
1	**0.1**	**0.2**
2	**0.3**	**0.4**

Answer:

X+Y	P(X+Y)
2	P(2)=P(X=1,Y=1)=0.1
3	P(3)=P(X=1,Y=2)+P(X=2,Y=1)=0.2+0.3=0.5
4	P(4)=P(X=2,Y=2)=0.4

$\therefore$

X+Y	2	3	4
Probability	0.1	0.5	0.4

15. The joint p.d.f of (X,Y) is given by $f(x,y)=6e^{-(x+y)},\ 0\le x,y\le\infty.$ **Are X and Y independent?**

Answer:

Marginal density of X:

$$f(x)=\int_{-\infty}^{\infty}f(x,y)dy=\int_0^{\infty}6e^{-(x+y)}dy=e^{-x},\ 0\le x$$

Marginal density of Y;

$$f(y)=\int_{-\infty}^{\infty}f(x,y)dx=\int_0^{\infty}6e^{-(x+y)}dx=e^{-y},\ y\le\infty$$

$$\Rightarrow f(x)f(y)=f(x,y)$$

$\therefore$ X and Y are independent.

16. The joint p.d.f of a bivariate R.V (X,Y) is given by

$$f(x,y)=\begin{cases}4xy, & 0<x<1, y<1\infty\\ 0 & ,otherwise\end{cases}$$ **.Find p(X+Y<1)**

Answer:

$$P[X+Y<1]=\int_0^1\int_0^{1-y} 4xydxdy = 2\int_0^1 y(1-y)^2 dy$$

$$=2\left[\frac{y^2}{2}-\frac{2y^3}{3}+\frac{y^4}{4}\right]_0^1$$

$$=2\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]=\frac{1}{6}$$

17. Define Co – Variance:

If X and Y are two two r.v.s then co – variance between them is defined as

Cov (X, Y) = E {X – E (X)} {Y – E (Y)}

(ie) Cov (X, Y) = E (XY) – E(X) E (Y)

18. State the properties of Co – variance;

1. If X and Y are two independent variables, then Cov (X,Y) = 0. But the Converse need not be true
2. Cov (aX, bY) = ab Cov (X,Y)
3. Cov (X + a,Y +b) = Cov (X,Y)
4. $Cov\left(\frac{X-\overline{X}}{\sigma_X},\frac{Y-\overline{Y}}{\sigma_Y}\right)=\frac{1}{\sigma_X\sigma_Y}Cov(X,Y)$
5. Cov(aX+b,cY+d)=acCov(X,Y)
6. Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z)
7. $Cov(aX+bY,cX+dY)=ac\sigma_X{}^2+bd\sigma_Y{}^2+(ad+bc)Cov(X,Y)$

$$where\,\sigma_X{}^2=Cov(X,X)=\operatorname{var}(X)\,and\,\sigma_Y{}^2=Cov(Y,Y)=\operatorname{var}(Y)$$

19.Show that Cov(aX+b,cY+d)=acCov(X,Y)

Answer:

Take U= aX+b and V= cY+d

Then E(U)=aE(X)+b and E(V)= cE(Y)+d

U-E(U)= a[X-E(X)] and V-E(V)=c[Y-E(Y)]

Cov(aX+b,cY+d)= Cov(U,V)=E[{U-E(U)}{V-E(V)}] = E[a{X-E(X)} c{Y-E(Y)}]

=ac E[{X-E(X)}{Y-E(Y)}]=acCov(X,Y)

20.If X&Y are independent R.V's ,what are the values of Var(X_1+X_2)and Var(X_1-X_2)

Answer:

$\mathrm{Var}(X_1\pm X_2)=\mathrm{Var}(X_1)+\mathrm{Var}(X_2)$ (Since X andY are independent RV then

$\mathrm{Var}(aX\pm bX)=a^2\mathrm{Var}(X)+b^2\mathrm{Var}(X)$)

21. If Y_1& Y_2 are independent R.V's ,then covariance (Y_1,Y_2)=0.Is the converse of the above statement true?Justify your answer.

Answer:

The converse is not true . Consider

$$X - N(0,1) \text{and } Y = X^2 \quad \sin ceX - N(0,1),$$

$$E(X)=0;\ E(X^3)=E(XY)=0 \sin ce\ all\ odd\ moments\quad vanish.$$

$$\therefore \operatorname{cov}(XY)=E(XY)-E(X)E(Y)=E(X^3)-E(X)E(Y)=0$$

$$\therefore \operatorname{cov}(XY)=0\quad but\quad X\ \&\ Y\quad are\,independent$$

22. Show that $\operatorname{cov}^2(X,Y)\le\operatorname{var}(X)\operatorname{var}(Y)$

Answer:

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

We know that $[E(XY)]^2 \le E(X^2)E(Y^2)$

$$\operatorname{cov}^2(X,Y) = [E(XY)]^2 + [E(X)]^2[E(Y)]^2 - 2E(XY)E(X)E(Y)$$
$$\le E(X)^2E(Y)^2 + [E(X)]^2[E(Y)]^2 - 2E(XY)E(X)E(Y)$$
$$\le E(X)^2E(Y)^2 + [E(X)]^2[E(Y)]^2 - E(X^2)E(Y)^2 - E(Y^2)E(X)^2$$
$$= \{E(X^2) - [E(X)]^2\}\{E(Y^2) - [E(Y)]^2\} \le \operatorname{var}(X)\operatorname{var}(Y)$$

23.If X and Y are independent random variable find covariance between X+Y and X-Y.

Answer:

$$\operatorname{cov}[X+Y, X-Y] = E[(X+Y)(X-Y)] - [E(X+Y)E(X-Y)]$$
$$= E[X^2] - E[Y^2] - [E(X)]^2 + [E(Y)]^2$$
$$= \operatorname{var}(X) - \operatorname{var}(Y)$$

24. X and Y are independent random variables with variances 2 and 3.Find the variance 3X+4Y.

Answer:

Given var(X) = 2, var(Y) = 3

We know that var(aX+Y) = a^2var(X) + var(Y)

And var(aX+bY) = a^2var(X) + b^2var(Y)

var(3X+4Y) = 3^2var(X) + 4^2var(Y) = 9(2) + 16(3) = 66

25. Define correlation

The correlation between two RVs X and Y is defined as

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(xy)\,dxdy$$

26. Define uncorrelated

Two RVs are uncorrelated with each other, if the correlation between X and Y is equal to the product of their means. i.e.,E[XY]=E[X].E[Y]

27. If the joint pdf of (X,Y) is given by $f(x,y) = e^{-(x+y)}, x \ge 0, y \ge 0.$ find E(XY).

Answer:

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dxdy = \int_0^{\infty}\int_0^{\infty} xye^{-(x+y)}\,dxdy = \int_0^{\infty} xe^{-x}dx\int_0^{\infty} ye^{-y}dy = 1$$

28. A R.V X is uniformly distributed over(-1,1) and Y=X^2. Check if X and Y are correlated?

Answer:

Given X is uniformly distributed in (-1,1),pdf of X is $f(x) = \dfrac{1}{b-a} = \dfrac{1}{2},\ -1 \le x \le 1$

$$E(X) = \frac{1}{2}\int_{-1}^{1} x\,dx = 0 \text{ and } E(XY) = E(X^3) = 0$$

$$\therefore \operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 0 \qquad \Rightarrow r(X,Y) = 0$$

Hence X and Y are uncorrelated.

29. X and Y are discrete random variables. If $\operatorname{var}(X) = \operatorname{var}(Y) = \sigma^2$,

$\operatorname{cov}(X,Y) = \dfrac{\sigma^2}{2}$, **find** $\operatorname{var}(2X - 3Y)$

Answer:

$$\text{var}(2X-3Y) = 4\,\text{var}(X) + 9\,\text{var}(Y) - 12\,\text{cov}(X,Y)$$

$$= 13\sigma^2 - 12\frac{\sigma^2}{2} = 7\sigma^2$$

30. If $\text{var}(X) = \text{var}(Y) = \sigma^2$, $\text{cov}(X,Y) = \frac{\sigma^2}{2}$, **find the correlation between** $2X+3$ **and** $2Y-3$

Answer:

$$r(aX+b, cY+d) = \frac{ac}{|ac|} r(X,Y) \ \ where\ a \neq 0, c \neq 0$$

$$\therefore r(2X+3, 2Y-3) = \frac{4}{|4|} r(X,Y) = r(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma^2/2}{\sigma.\sigma} = \frac{1}{2}$$

31. Two independent random variables X and Y have 36 and 16.Find the correlation co-efficient between X+Y and X-Y

Answer:

$$\therefore r(X+Y, X-Y) = \frac{\sigma_X{}^2 - \sigma_Y{}^2}{\sigma_X{}^2 + \sigma_Y{}^2} = \frac{36-16}{36+16} = \frac{20}{52} = \frac{4}{13}$$

32. If the lines of regression of Y on X and X on Y are respectively $a_1X + b_1Y + c_1 = 0$ **and** $a_2X + b_2Y + c_2 = 0$ **,prove that** $a_1b_2 \leq a_2b_1$ **.**

Answer:

$$b_{yx} = -\frac{a_1}{b_1} \qquad and \qquad b_{xy} = -\frac{b_2}{a_2}$$

Since $r^2 = b_{yx}b_{yx} \leq 1 \quad \Rightarrow \frac{a_1}{b_1}.\frac{b_2}{a_2} \leq 1$

$$\therefore a_1b_2 \leq a_2b_1$$

33. State the equations of the two regression lines. what is the angle between them?

Answer:

Regression lines:

$$y - \bar{y} = r\frac{\sigma y}{\sigma x}(x - \bar{x}) \ \ and\ \ x - \bar{x} = r\frac{\sigma x}{\sigma y}(y - \bar{y})$$

Angle $\theta = \tan^{-1}\left[\frac{1-r^2}{r}\left(\frac{\sigma_x \sigma_y}{\sigma_x{}^2 + \sigma_y{}^2}\right)\right]$

34. The regression lines between two random variables X and Y is given by $3X + Y = 10$ **and** $3X + 4Y = 12$ **.Find the correlation between X and Y.**

Answer:

$$3X + 4Y = 12 \qquad \Rightarrow b_{yx} = -\frac{3}{4}$$

$$3X + Y = 10 \qquad \Rightarrow b_{xy} = -\frac{1}{3}$$

$$r^2 = \left(-\frac{3}{4}\right)\left(-\frac{1}{3}\right) = \frac{1}{4} \qquad \Rightarrow r = -\frac{1}{2}$$

35. Distinguish between correlation and regression.

Answer:

By correlation we mean the casual relationship between two or more variables.
By regression we mean the average relationship between two or more variables.

36. State the Central Limit Theorem.

Answer:

If $x_1, x_2, ..., x_n$ are n independent identically distributed RVs with men μ and S.D σ and if $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, then the variate $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$ has a distribution that approaches the standard normal distribution as $n \to \infty$ provided the m.g.f of x_i exist.

37. The lifetime of a certain kind of electric bulb may be considered as a RV with mean 1200 hours and S.D 250 hours. Find the probability that the average life time of exceeds 1250 hours using central limit theorem.

Solution:

Let X denote the life time of the 60 bulbs.

Then μ = E(X)= 1200 hrs. and Var(X)=(S.D)^2= σ^2=$(250)^2$hrs.

Let $\overline{X}$ denote the average life time of 60 bulbs.

By Central Limit Theorem, $\overline{X}$ follows $N\left(\mu, \frac{\sigma^2}{n}\right)$.

Let $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$ be the standard normal variable

$$P[\overline{X} > 1250] = P[Z > 1.55]$$
$$= 0.5 - P[0 < Z < 1.55]$$
$$= 0.5 - 0.4394 = 0.0606$$

38.Joint probability distribution of (X, Y)

Let (X,Y) be a two dimensional discrete random variable. Let $P(X=x_i, Y=y_j)=p_{ij}$. p_{ij} is called the probability function of (X,Y) or joint probability distribution. If the following conditions are satisfied

1.$p_{ij} \geq 0$ for all i and j

2.$\sum_j \sum_i p_{ij} = 1$

The set of triples (x_i, y_j, p_{ij}) i=1,2,3……. And j=1,2,3……is called the Joint probability distribution of (X,Y)

39.Joint probability density function

If (X,Y) is a two-dimensional continuous RV such that

$$P\left\{x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2} and y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\right\} = f(x, y)dxdy$$

Then f(x,y) is called the joint pdf of (X,Y) provided the following conditions satisfied.

1. $f(x, y) \geq 0 \, for \ all \ (x, y) \in (-\infty, \infty)$

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1 \ and \ f(x, y) \geq 0 \, for \, all (x, y) \in (-\infty, \infty)$

40.Joint cumulative distribution function (joint cdf)

If (X,Y) is a two dimensional RV then $F(x, y) = P(X \leq x, Y \leq y)$ is called joint cdf of (X,Y)

In the discrete case,

$$F(x, y) = \sum_{\substack{j \\ y_j \leq y}} \sum_{\substack{i \\ x_i \leq x}} p_{ij}$$

In the continuous case,

$$F(x, y) = P(-\infty < X \leq x, -\infty < Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x, y)dxdy$$

41.Marginal probability distribution(Discrete case)

Let (X,Y) be a two dimensional discrete RV and $p_{ij}=P(X=x_i,Y=y_j)$ then

$$P(X=x_i)=p_{i*}=\sum_j p_{ij}$$

is called the Marginal probability function.

The collection of pairs $\{x_i,p_{i*}\}$ is called the Marginal probability distribution of X.

If $P(Y=y_j)=p_{*j}=\sum_i p_{ij}$ then the collection of pairs $\{x_i,p_{*j}\}$ is called the Marginal probability distribution of Y.

42.Marginal density function (Continuous case)

Let f(x,y) be the joint pdf of a continuous two dimensional RV(X,Y).The marginal density function of X is defined by $f(x)=\int_{-\infty}^{\infty} f(x,y)dy$

The marginal density function of Y is defined by $f(y)=\int_{-\infty}^{\infty} f(x,y)dx$

43.Conditional probability function

If $p_{ij}=P(X=x_i,Y=y_j)$ is the Joint probability function of a two dimensional discrete RV(X,Y) then the conditional probability function X given $Y=y_j$ is defined by

$$P\left[X=x_i / Y=y_j\right]=\frac{P\left[X=x_i \cap Y=y_j\right]}{P\left[Y=y_j\right]}$$

The conditional probability function Y given $X=x_i$ is defined by

$$P\left[Y=y_j / X=x_i\right]=\frac{P\left[X=x_i \cap Y=y_j\right]}{P\left[X=x_i\right]}$$

44.Conditional density function

Let f(x,y) be the joint pdf of a continuous two dimensional RV(X,Y).Then the Conditional density function of X given Y=y is defined by $f(X/Y)=\frac{f_{XY}(x,y)}{f_Y(y)}$,where f(y)=marginal p.d.f of Y.

The Conditional density function of Y given X=x is defined by $f(Y/X)=\frac{f_{XY}(x,y)}{f_X(x)}$, where f(x)=marginal p.d.f of X.

45.Define statistical properties

Two jointly distributed RVs X and Y are statistical independent of each other if and only if the joint probability density function equals the product of the two marginal probability density function

i.e., f(x,y)=f(x).f(y)

46.The joint p.d.f of (X,Y) is given by $f(x,y)=e^{-(x+y)}$ $0\le x,y\le\infty$ .Are X and Y are independent?

Answer:

Marginal densities:

f(x)= $\int_0^{\infty} e^{-(x+y)}dy=e^{-x}$ and f(y)= $\int_0^{\infty} e^{-(x+y)}dx=e^{-y}$

X and Y are independent since f(x,y)=f(x).f(y)

47.Define Co – Variance :

If X and Y are two two r.v.s then co – variance between them is defined as

Cov (X, Y) = E {X – E (X)} {Y – E (Y)}

(ie) Cov (X, Y) = E (XY) – E(X) E (Y)

48. State the properties of Co – variance;

1. If X and Y are two independent variables, then Cov (X,Y) = 0. But the Converse need not be true
2. Cov (aX, bY) = ab Cov (X,Y)
3. Cov (X + a,Y +b) = Cov (X,Y)
4. $Cov\left(\frac{X-\overline{X}}{\sigma_X},\frac{Y-\overline{Y}}{\sigma_Y}\right)=\frac{1}{\sigma_X\sigma_Y}Cov(X,Y)$
5. Cov(aX+b,cY+d)=acCov(X,Y)
6. Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z)
7. $Cov(aX+bY,cX+dY)=ac\sigma_X{}^2+bd\sigma_Y{}^2+(ad+bc)Cov(X,Y)$

$$where\,\sigma_X{}^2=Cov(X,X)=\mathrm{var}(X)\,and\,\sigma_Y{}^2=Cov(Y,Y)=\mathrm{var}(Y)$$

49.Show that Cov(aX+b,cY+d)=acCov(X,Y)

Answer:

Take U= aX+b and V= cY+d

Then E(U)=aE(X)+b and E(V)= cE(Y)+d

U-E(U)= a[X-E(X)] and V-E(V)=c[Y-E(Y)]

Cov(aX+b,cY+d)= Cov(U,V)=E[{U-E(U)}{V-E(V)}] = E[a{X-E(X)} c{Y-E(Y)}]

=ac E[{X-E(X)}{Y-E(Y)}]=acCov(X,Y)

Unit III MarkovProcesses and Markov chains

1. Define Random processes and give an example of a random process.

Answer:

A Random process is a collection of R.V $\{X(s,t)\}$ that are functions of a real variable namely time t where $s \in S$ and $t \in T$

Example: $X(t) = A\cos(\omega t + \theta)$ where θ is uniformly distributed in $(0, 2\pi)$ where A and ω are constants.

2. State the four classifications of Random processes.

Sol: The Random processes is classified into four types

(i)Discrete random sequence

If both T and S are discrete then Random processes is called a discrete Random sequence.

(ii)Discrete random processes

If T is continuous and S is discrete then Random processes is called a Discrete Random processes.

(iii)Continuous random sequence

If T is discrete and S is continuous then Random processes is called a Continuous Random sequence.

(iv)Continuous random processes

If T &S are continuous then Random processes is called a continuous Random processes.

3. Define stationary Random processes.

If certain probability distributions or averages do not depend on t, then the random process $\{X(t)\}$ is called stationary.

4. Define first order stationary Random processes.

A random processes $\{X(t)\}$ is said to be a first order SSS process if $f(x_1, t_1 + \delta) = f(x_1, t_1)$ (i.e.) the first order density of a stationary process $\{X(t)\}$ is independent of time t

5. Define second order stationary Random processes

A RP $\{X(t)\}$ is said to be second order SSS if $f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1 + h, t_2 + h)$ where $f(x_1, x_2, t_1, t_2)$ is the joint PDF of $\{X(t_1), X(t_2)\}$.

6. Define strict sense stationary Random processes

Sol: A RP $\{X(t)\}$ is called a SSS process if the joint distribution $X(t_1)X(t_{21})X(t_3).........X(t_n)$ is the same as that of $X(t_1 + h)X(t_2 + h)X(t_3 + h).........X(t_n + h)$ for all $t_1, t_2, t_3.........t_n$ and h >0 and for $n \geq 1$.

7. Define wide sense stationary Random processes

A RP $\{X(t)\}$ is called WSS if $E\{X(t)\}$ is constant and $E[X(t)X(t+\tau)] = R_{xx}(\tau)$ (i.e.) ACF is a function of τ only.

8. Define jointly strict sense stationary Random processes

Sol: Two real valued Random Processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the strict sense if the joint distribution of the $\{X(t)\}$ and $\{Y(t)\}$ are invariant under translation of time.

9. Define jointly wide sense stationary Random processes

Sol: Two real valued Random Processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the wide sense if each process is individually a WSS process and $R_{XY}(t_1, t_2)$ is a function of t_1, t_2 only.

10. Define Evolutionary Random processes and give an example.

Sol: A Random processes that is not stationary in any sense is called an Evolutionary process. Example: Poisson process.

11. If $\{X(t)\}$ is a WSS with auto correlation $R(\tau) = Ae^{-\alpha|\tau|}$, determine the second order moment of the random variable X (8) – X (5).

Sol: Given $R_{xx}(\tau) = Ae^{-\alpha|\tau|}$ (i.e.) $R_{xx}(t_1, t_2) = Ae^{-\alpha|t_1 - t_2|}$

(i.e.) $E(X(t_1).X(t_2)) = Ae^{-\alpha|t_1 - t_2|}$ ………. (1)

$\therefore E\left(X^2(t)\right) = E(X(t)X(t)) = R_{xx}(t,t) = Ae^{-\alpha(0)} = A$

$\therefore E(X^2(8)) = A \;\&\; E(X^2(5)) = A \;\; \therefore E(X(8)X(5)) = R_{xx}(8,5) = A.e^{-3\alpha}$.

Now second order moment of $\{X(8) - X(5)\}$ is given by

$$E(X(8) - X(5))^2 = E(X^2(8) + X^2(5) - 2X(8)X(5))$$
$$= E(X^2(8)) + E(X^2(5)) - 2E(X(8)X(5))$$
$$= A + A - 2Ae^{-3\alpha} = 2A(1 - e^{-3\alpha})$$

12. Verify whether the sine wave process $\{X(t)\}$, where $X(t) = Y\cos\omega t$ where Y is uniformly distributed in (0,1) is a SSS process.

Sol: $F(x) = P(X(t) \le x) = P(Y\cos\omega t \le x)$

$$= \begin{cases} P\left(Y \le \dfrac{x}{\cos\omega t}\right) if \cos\omega t > 0 \\ P\left(Y \ge \dfrac{x}{\cos\omega t}\right) if \cos\omega t < 0 \end{cases}$$

$$F_X(x) = \begin{cases} F_Y\left(\dfrac{x}{\cos\omega t}\right) if \cos\omega t > 0 \\ 1 - F_Y\left(\dfrac{x}{\cos\omega t}\right) if \cos\omega t < 0 \end{cases}$$

$$\therefore f_{X(t)}(x) = \frac{1}{|\cos\omega t|} f_Y\left(\frac{x}{\cos\omega t}\right) = \text{a function of t}$$

If {X(t)} is to be a SSS process, its first order density must be independent of t. Therefore, {X(t)} is not a SSS process.

13. Consider a random variable $Z(t) = X_1 \cos\omega_0 t - X_2 \sin\omega_0 t$ where X_1 and X_2 are independent Gaussian random variables with zero mean and variance σ^2 find E(Z) and E(Z^2)

Sol: Given $E(X_1) = 0 = E(X_2)$ & $Var(X_1) = \sigma^2 = Var(X_2)$

$$\Rightarrow E(X_1^{\,2}) = \sigma^2 = E(X_2^{\,2})$$

$$E(Z) = E(X_1 \cos\omega_0 t - X_2 \sin\omega_0 t) = 0$$

$$E(Z^2) = E(X_1 \cos\omega_0 t - X_2 \sin\omega_0 t)^2$$

$$= E(X_1^{\,2})\cos^2\omega_0 t + E(X_2^{\,2})\sin^2\omega_0 t - E(X_1 X_2)\cos\omega_0 t \sin\omega_0 t$$

$$= \sigma^2(\cos^2\omega t + \sin^2\omega t) - E(X_1)E(X_2)\cos\omega_0 t \sin\omega_0 t \qquad \because X_1 \,\&\, X_2 \text{ are independent}$$

$$= \sigma^2 - 0 = \sigma^2.$$

14. Consider the random process $X(t) = \cos(\omega_0 t + \theta)$ where θ is uniformly distributed in $(-\pi, \pi)$ .Check whether X(t) is stationary or not?

Answer:

$$E[X(t)] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega_0 t + \theta)d\theta = \frac{1}{2\pi}[\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)] = \frac{1}{2\pi}[-\sin(\omega_0 t) + \sin(\omega_0 t)] = 0$$

$$E[X^2(t)] = \frac{1}{4\pi}\left[\frac{\theta - 2\sin(\omega_0 t + \theta)}{2}\right]_{-\pi}^{\pi} = \frac{1}{2}$$

15. Define Markov Process.

Sol: If for $t_1 < t_2 < t_3 < t_4 \ldots\ldots\ldots\ldots < t_n < t$ then

$$P(X(t) \le x / X(t_1) = x_1, X(t_2) = x_2, \ldots\ldots\ldots\ldots X(t_n) = x_n) = P(X(t) \le x / X(t_n) = x_n)$$

Then the process $\{X(t)\}$ is called a Markov process.

16. Define Markov chain.

Sol: A Discrete parameter Markov process is called Markov chain.

17. Define one step transition probability.

Sol: The one step probability $P\left[X_n = a_j / X_{n-1} = a_i\right]$ is called the one step probability from the state a_i to a_j at the n^{th} step and is denoted by $P_{ij}(n-1, n)$

18. The one step tpm of a Markov chain with states 0 and 1 is given as $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .Draw the Transition diagram. Is it Irreducible Markov chain?

Sol:

Yes it is irreducible since each state can be reached from any other state

.

19. .Prove that the matrix P = $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ **is the tpm of an irreducible Markov chain.**

Sol: $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ $P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

Here $P_{11}^{(3)} > 0.P_{13}^{(2)} > 0, P_{21}^{(2)} > 0, P_{22}^{(2)} > 0, P_{33}^{(2)} > 0$ and for all other $P_{ij}^{(1)} > 0$
Therefore the chain is irreducible.

20 State the postulates of a Poisson process.

Let $\{X(t)\}$ = number of times an event A say, occurred up to time 't' so that the sequence $\{X(t)\}$, $t \geq 0$ forms a Poisson process with parameter λ.

(i) P[1 occurrence in $(t, t+\Delta t)$]= $\lambda \Delta t$

(ii) P[0 occurrence in $(t, t+\Delta t)$]=1- $\lambda \Delta t$

(iii) P[2 or more occurrence in $(t, t+\Delta t)$]=0

(iv) X(t) is independent of the number of occurrences of the event in any interval prior and after the interval (0,t).

(v) The probability that the event occurs a specified number of times in (t_0, t_0+t) depends only on t, but not on t_0.

20. State any two properties of Poisson process

Sol: (i) The Poisson process is a Markov process

(ii) Sum of two independent Poisson processes is a Poisson process

(iii) The difference of two independent Poisson processes is not a Poisson process.

21. If the customers arrived at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between two consecutive arrivals is more than one minute.

Sol: The interval T between 2 consecutive arrivals follows an exponential distribution with

parameter $\lambda = 2, P(T > 1) = \int_1^\infty 2e^{-2t} dt = e^{-2} = 0.135.$

22. A bank receives on an average $\lambda = 6$ bad checks per day, what are the probabilities that it will receive (i) 4 bad checks on any given day (ii) 10 bad checks over any 2 consecutive days.

Sol: $P(X(t) = n) = \frac{e^{-\lambda t}.(\lambda t)^n}{n!} = \frac{e^{-6t}(6t)^n}{n!}, n = 0,1,2....$

(i) $P(X(1) = 4) = \frac{e^{-6}(6)^4}{4!} = 0.1338$

(ii) $P(X(2) = 10) = \frac{e^{-12}(12)^{10}}{10!} = 0.1048$

23. Suppose the customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. Find the probability that during a time interval of 2 minutes exactly 4 customers arrive

Sol: (i) $P(X(t)=n)=\dfrac{e^{-3t}(3t)^n}{n!}, n=0,1,2.....$

(ii) $P(X(2)=4)=\dfrac{e^{-6}(6)^4}{4!}=0.1338$.

24. Consider a Markov chain with two states and transition probability matrix $P=\begin{bmatrix}3/4 & 1/4\\ 1/2 & 1/2\end{bmatrix}$.Find the stationary probabilities of the chain.

Sol: $(\pi_1,\pi_2)\begin{bmatrix}3/4 & 1/4\\ 1/2 & 1/2\end{bmatrix}=(\pi_1,\pi_2) \qquad \pi_1+\pi_2=1$

$\dfrac{3}{4}\pi_1+\dfrac{\pi_2}{4}=\pi_1 \Rightarrow \dfrac{\pi_1}{4}-\dfrac{\pi_2}{2}=0. \qquad \therefore \pi_1=2\pi_2$

$\therefore \pi_1=\dfrac{2}{3},\pi_2=\dfrac{1}{3}.$

25. Customers arrive a large store randomly at an average rate of 240 per hour. What is the probability that during a two-minute interval no one will arrive.

Sol: $P(X(t)=n)=\dfrac{e^{-4t}.(4t)^n}{n!}, n=0,1,2....$ since $\lambda=\dfrac{240}{60}=4$

$\therefore P(X(2)=0)=e^{-8}=0.0003.$

26. The no of arrivals at the reginal computer centre at express service counter between 12 noon and 3 p.m has a Poison distribution with a mean of 1.2 per minute. Find the probability of no arrivals during a given 1-minute interval.

Sol: $P(X(t)=n)=\dfrac{e^{-1.2t}.(1.2t)^n}{n!}, n=0,1,2....$

$P(X(1)=0)=e^{-1.2}=0.3012.$

Sol: (i) If a Gaussian process is wide sense stationary, it is also a strict sense stationary.

$R_{yy}(\tau)=\dfrac{2}{\pi}R_{xx}(0)[\cos\alpha+\alpha\sin\alpha]$ where $\sin\alpha=\dfrac{R_{xx}(\tau)}{R_{xx}(0)}$.

Hence $\{Y(t)\}$ is wide sense stationary.

Then $Y(t)$ is called a Hard limiter process or ideal limiter process.

27. For the sine wave process $X(t)=Y\cos\omega t, -\infty<t<\infty$ where ω= constant, the amplitude Y is a random variable with uniform distribution in the interval 0 and 1.check wheather the process is stationary or not.

Sol: $f(y)=\begin{cases}1 & 0<y<1\\ 0 & \text{Otherwise}\end{cases}$

$E(X(t))=\int_0^1 1.Y\cos\omega t=\cos\omega t\int_0^1 y dy=\cos\omega t.$(a function of t)

Therefore it is not stationary.

28. Derive the Auto Correlation of Poisson Process.

Sol: $R_{xx}(t_1,t_2)=E[X(t_1)X(t_2)]$

$R_{xx}(t_1,t_2)=E[X(t_1)\{X(t_2)-X(t_1)+X(t_1)\}]$

$=E[X(t_1)\{X(t_2)-X(t_1)\}]+E[X^2(t_1)]$

$=E[X(t_1)]E[X(t_2)-X(t_1)]+E[X^2(t_1)]$

Since $X(t)$ is a Poisson process, a process of independent increments.

$\therefore R_{xx}(t_1,t_2)=\lambda t_1(\lambda t_2-\lambda t_1)+\lambda_1 t_1+\lambda_1^2 t_1^2$ if $t_2\ge t_1$

$$\Rightarrow R_{xx}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda t_1 \text{ if } t_2 \geq t_1$$

$$\text{(or) } \Rightarrow R_{xx}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min\{t_1,t_2\}$$

29. Derive the Auto Covariance of Poisson process

Sol: $C(t_1,t_2) = R(t_1,t_2) - E[X(t_1)]E[X(t_2)]$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2 = \lambda t_1 \text{ if } t_2 \geq t_1$$

$$\therefore C(t_1,t_2) = \lambda \min\{t_1,t_2\}$$

30. Define Time averages of Random process.

Sol: The time averaged mean of a sample function $X(t)$ of a random process $\{X(t)\}$ is defined as $\overline{X_T} = \lim_{t\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt$

The time averaged auto correlation of the Random process $\{X(t)\}$ is defined by $\overline{Z_T} = \lim_{t\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)dt$.

31. If {X(t)} is a Gaussian process with $\mu(t) = 10$ and $C(t_1,t_2) = 16e^{-|t_1-t_2|}$, Find P{X(10)≤8}.

Answer: $\mu[X(10)] = 10$ *and* $Var[X(10)] = C(10,10) = 16$

$$P[X(10) \leq 8] = P\left[\frac{X(10)-10}{4} \leq -0.5\right] = P[Z \leq -0.5] = 0.5 - P[Z \leq 0.5] = 0.5 - 0.1915 = 0.3085$$

32. If {X(t)} is a Gaussian process with $\mu(t) = 10$ and $C(t_1,t_2) = 16e^{-|t_1-t_2|}$, Find the mean and variance of X(10)-X(6).

Answer:

X(10)-X(6) is also a normal R.V with mean $\mu(10) - \mu(6) = 0$ *and*

$$Var[X(10) - X(6)] = \text{var}\{X(10)\} + \text{var}\{X(6)\} - 2\,\text{cov}\{X(10), X(6)\}$$

$$= C(10,10) + C(6,6) - 2C(10,6) = 16 + 16 - 2\times 16e^{-4} = 31.4139$$

33.Define a Birth process.

Answer:

A Markov process {X(t)} with state space S={1,2…..} such that

$$P[X(t+s_t) = k / X(t) = j] = \begin{cases} \lambda_k s_t, k = j+1, j \geq 1 \\ 1-\lambda, s_t k = j \geq 1 \\ 0, otherwise \end{cases}$$

is called a birth process where $\lambda_1, \lambda_2, ...$ are the constant.

34.*Define Ergodic Random Process*.

Sol: A random process $\{X(t)\}$ is said to be Ergodic Random Processif its ensembled averages
are equal to appropriate time averages.

35.Define Ergodic state of a Markov chain.

Sol: A non null persistent and aperiodic state is called an ergodic state.

36.Define Absorbing state of a Markov chain.

Sol: A state i is called an absorbing state if and only if $P_{ij} = 1$ and $P_{ij} = 0$ for $i \neq j$

37.Define irreducible

The process is stationary as the first and second moments are independent of time.
State any four properties of Autocorrelation function.
Answer:
i) $R_{XX}(-\tau) = R_{XX}(\tau)$
ii) $|R(\tau)| \leq R(0)$
iii) $R(\tau)$ is continuous for all τ
iv) if $R_{XX}(-\tau)$ is AGF of a stationary random process {X(t)} with no periodic components, then $\mu_X{}^2 = \lim_{\tau \to \infty} R(\tau)$

38.What do you mean by absorbing Markov chain? Give an example.
Sol: A State I of a Markov chain is said to be an absorbing state if $P_{ii} = 1$ (i.e.) it is impossible to leave it .A Markov chain is said to be absorbing if it has at least one absorbing state.
Eg: The tpm of an absorbing Markov chain is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

39.Define Bernoulli Process.
Sol: The Bernoulli random variable is defined as $X(t_i) = t_i$ which takes two values 0 and 1 with the time index t_i such that $\{X(t_i, s) : i = ...-1,0,1,......; s = 0,1\}$
40State the properties of Bernoulli Process.
Sol: (i) It is a Discrete process
(ii) It is a SSS process
(iii) $E(X_i) = p, E(X^2) = p$ and $Var(X_i) = p(1-p)$
41.Define Binomial Process
Sol: It is defined as a sequence of partial sums $\{S_n\}, n = 1,2,3.....$ where
$S_n = X_1 + X_2 + ... + X_n$
42.State the Basic assumptions of Binomial process
Sol: (i) The time is assumed to be divided into unit intervals. Hence it is a discrete time process.
(ii) At most one arrival can occur in any interval
(iii) Arrivals can occur randomly and independently in each interval with probability p.
43.Prove that Binomial process is a Markov process
Sol: $S_n = X_1 + X_2 + ... + X_n$
Then $S_n = S_{n-1} + X_n$
$\therefore P(S_n = m / S_{n-1} = m) = P(X_n = 0) = 1 - p.$
Also $P(S_n = m / S_{n-1} = m - 1) = P(X_n = 1) = p.$
(i.e.) the probability distribution of S_n ,depends only on S_{n-1}.The process is Markovian process.
44. Define Sine wave process.
Sol: A sine wave process is represented as $X(t) = A\sin(\omega t + \theta)$ where the amplitude A ,or frequency ω or phase θ or any combination of these three may be

random.

It is also represented as $X(t) = A\cos(\omega t + \theta)$.

45.Prove that sum of two independent Poisson process is also Poisson.

Sol: Let $X(t) = X_1(t) + X_2(t)$

$$P[X(t) = n] = \sum_{k=0}^{n} P[X_1(t) = k] P[X_2(t) = n-k]$$

$$= \sum_{k=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_2+\lambda_2)t}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (\lambda_1 t)^k (\lambda_2 t)^{n-k}$$

$$= \frac{e^{-(\lambda_2+\lambda_2)t}}{n!} (\lambda_1 t + \lambda_2 t)^n$$

Therefore $X(t) = X_1(t) + X_2(t)$ is a Poisson process with parameter $(\lambda_1 + \lambda_2)t$

Unit IV-Queueing Theory

1) What is meant by queue Discipline ?

Answer:

It Specifies the manner in which the customers from the queue or equivalently the manner in which they are selected for service, when a queue has been formed. The most common discipline are

(i) FCFS (First Come First Served) or First In First Out (FIFO)

(ii) LCFS (Last Come First Served)

(iii) SIRO (Service in Random Order)

2) Define Little's formula

Answer:

(i) $L_s = \lambda W_s$ (ii) $L_q = \lambda W_q$ (iii) $W_5 = W_q + \frac{1}{\mu}$ (iv) $L_s = L_q + \frac{\lambda}{\mu}$

3) If people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 minutes before the picture starts and if it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket. Can he expect to be seated for the start of the picture ?

Answer:

Here = $\lambda = 6,\ \frac{1}{\mu} = \frac{15}{2} \Rightarrow \mu = \frac{2}{15}/\sec$

$\therefore \lambda = 6;\quad \mu = \frac{2}{15}\ x\ 60 = 8$

$E(L_q) = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2}\min$

He can just be seated for the start of the picture.

$\therefore$ E [Total time] = $\frac{1}{2} + \frac{3}{2} = 2\min$

4) If λ, μ are the rates of arrival and departure in a M/M/I queue respectively, give the formula for the probability that there are n customers in the queue at any time in steady state.

Answer:

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left[1 - \frac{\lambda}{\mu}\right]$$

5) If λ, μ are the rates of arrival and departure respectively in a M/M/I queue, write the formulas for the average waiting time of a customer in the queue and the average number of customers in the queue in the steady state.

Answer:

$$E[N_q] = \frac{\lambda^2}{\mu(\mu - \lambda)}, E[W_q] = \frac{\lambda}{\mu(\mu - \lambda)}$$

6) If the arrival and departure rates in a public telephone booth with a single phone are $\frac{1}{12}$ *and* $\frac{1}{4}$ respectively, find the probability that the phone is busy.

Answer:

P[Phone is busy] = 1 – P [No customer in the booth]

$$= 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = \frac{1}{12}/\frac{1}{4} = \frac{1}{3}$$

7) If the inter-arrival time and service time in a public telephone booth with a single-phone follow exponential distributions with means of 10 and 8 minutes respectively, Find the average number of callers in the booth at any time.

Answer:

$L_s = \frac{\lambda}{\mu - \lambda}$ Here $\lambda = \frac{1}{10}$ and $\mu = \frac{1}{8}$

$\therefore$ Average no. of callers in the booth $= \dfrac{1/10}{\frac{1}{8} - \frac{1}{10}} = \frac{1}{10} x \frac{40}{1} = 4$

8) If the arrival and departure rates in a M/M/I queue are $\frac{1}{2}$ per minute and $\frac{2}{3}$ per minute respectively, find the average waiting time of a customer in the queue.

Answer:

Average waiting time of a customer in the queue

$$E(W_q) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/2}{\frac{2}{3}\left(\frac{2}{3}-\frac{1}{2}\right)} = 4.5\,\text{min}$$

9) Customers arrive at a railway ticket counter at the rate of 30/hour. Assuming Poisson arrivals, exponential service time distribution and a single server queue (M/M/I) model, find the average waiting time (before being served) if the average service time is 100 seconds.

Answer:

λ = 30/hour and $\frac{1}{\mu}$ = 100 sec $\Rightarrow$ $\frac{1}{\mu} = \frac{1}{36}$ hour $\therefore$ = 36 / hour

Average waiting time in the queue

$$E[W_q] = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{30}{36x6} = \frac{5}{36} hour = 8.33\,\text{min}$$

10) What is the probability that a customer has to wait more than 15 minutes to get his service completed in (M/M/I) : (∞ / FIFO) queue systme if λ = 6 per hour and μ = 10 per hour ?

Answer:

probability that the waiting time of a customer in the system exceeds time t=$e^{-(\mu-\lambda)t}$

Here λ=6, μ = 10, t = 15 min = $\frac{1}{4}$ hour

$\therefore$ Required probability = $e^{-(10-6)\frac{1}{4}} = e^{-1} = \frac{1}{e} = 0.3679$

11) For (M/M/I) : (∞ / FIFO) model, write down the Little's formula.

Answer:

(i) $E(N_s) = \lambda E(W_s)$ (ii) $E(N_q) = \lambda E(W_q)$

(iii) $E(W_s) = E(W_q) + \frac{1}{\mu}$ (iv) $E(N_s) = E(N_q) + \frac{\lambda}{\mu}$

12) Using the Little's formula, obtain the average waiting time in the system for M|M|1|N model.

Answer:

By the modified Little's formula,

$E(W_S) = \frac{1}{\lambda'}$ E (N_S) where λ' is the effective arrival rate.

13) For (M/M/I) : (∞ / FIFO) model, write down the formula for

a. **Average number of customers in the queue**

b. **Average waiting time in the system.**

Answer:

(i) $E(W_q)$ or $L_q = \frac{P_0(\rho C)^C}{C!(1-\rho)^2}\left[1-\rho^{N-C}-(1-\rho)(N-C)\rho^{N-C}\right]$

Where $\rho = \frac{\lambda}{C\mu} and P_0 = \left[\sum_{n=0}^{C-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{N}\frac{1}{C^{n-c}}\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$

(ii) $E(W_s) = \frac{L_S}{\lambda'}$ where $\lambda' = \lambda(1-P_N)$ or $\mu\left[C-\sum_{n=0}^{C-1}(C-n)P_n\right]$

and $L_S = L_q + C - \sum_{0}^{C-1}(C-n)P_n$

14) What is the probability that a customer has to wait more than 15 minutes to get his service completed in M|M|1 queuing system, if λ = 6 per hour and μ = 10 per hour ?

Answer:

Probability that the waiting time in the system exceeds t is

$$\int_t^{\infty}(\mu-\lambda)e^{-(\mu-\lambda)\varpi}d\varpi = e^{-(\mu-\lambda)t}$$

$$P\left[W_S > \frac{15}{60}\right] = e^{-(10-6)\frac{1}{4}} = e^{-1} = 0.3679$$

15) What is the probability that an arrival to an infinite capacity 3 server Poisson queuing system with $\frac{\lambda}{\mu} = 2$ and $p_0 = \frac{1}{9}$ enters the service with out waiting ?

Answer:

P [with out waiting] = P [N < 3] = $P_0 + P_1 + P_2$

$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$ When n ≤ C. Here C = 3

$$\therefore [N < 3] = \frac{1}{9} + \frac{2}{9} + \frac{1}{2} X \frac{4}{9} = \frac{5}{9}$$

16) Consider an M|M|1 queuing system. If λ = 6 and μ = 8, Find the probability of atleast 10 customers in the system.

Answer:

$$P[n \geq 10] = \sum_{n=10}^{\infty} P_n = \sum_{n=10}^{\infty} P_n \left(1 - \frac{6}{8}\right)\left(\frac{6}{8}\right)^{10} - \left(\frac{6}{8}\right)^{10} = \left(\frac{3}{4}\right)^{10} \; (k = 9)$$

$$\left[\text{Probability that the number of customers in the sytem exceeds k, P}\left(\text{n} > \text{K}\right) = \left(\frac{\lambda}{\mu}\right)^{k+1}\right]$$

17) Consider an M|M|C queuing system. Find the probability that an arriving customer is forced to join the queue.

Answer:

$$\text{P}\left[\text{a customer is forced to join queue}\right] = \sum_{\text{n = c}}^{\infty} \text{P}_\text{n}$$

$$= P_0 \frac{(C\rho)^c}{C!(1-\rho)} = \frac{\frac{(C\rho)^c}{C!(1-\rho)}}{\sum_{n=0}^{C-1} \frac{(C\rho)^n}{n!} + \frac{(C\rho)^c}{C!(1-\rho)}} \quad \text{Where } \rho = \frac{\lambda}{C\mu}$$

18) Write down Pollaczek-Khinchine formulae.

Answer:

(i) Average number of customers in the system = $\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho$

(ii) Average queue length = $\frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$ Where $\rho = \lambda E(T)$; $\sigma^2 = V(T)$

19) Consider an M|M|1 queueing system. Find the probability of finding atleast 'n' customers in the system.

Answer:

Probability of at least n customers in the system

$$P[N \geq n] = \sum_{K=n}^{\infty} P_k = \sum_{K=n}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right)$$

$$\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \sum_{k=n}^{\infty} \left(\frac{\lambda}{\mu}\right)^{k-n} = \left(\frac{\lambda}{\varpi}\right)^n \left[1 - \frac{\lambda}{\mu}\right]\left[1 - \frac{\lambda}{\mu}\right]^{-1} = \left(\frac{\lambda}{\mu}\right)^n$$

20) Consider an M|M|C queueing system. Find the probability that an arriving customer is forced to join the queue.

Answer:

$$P[N \geq C] = \sum_{n=c}^{\infty} P_n = \sum_{n=c}^{\infty} \frac{1}{C!C^{n-c}} \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^c P_0 \sum_{n=c}^{\infty} \left(\frac{\lambda}{C\mu}\right)^{n-c} = \frac{\left(\frac{\lambda}{\mu}\right)^c P_0}{C!\left(1 - \frac{\lambda}{C\mu}\right)}$$

21) Briefly describe the M|G|1 queuing system.

Answer:

Poisson arrival / General service / Single server queuing system.

22) Arrivals at a telephone booth are considered to be Poisson with an average time of 12 minutes between one arrival and the next. The length of a phone call is assumed to be exponentially distributed with mean 4 min. What is the probability that it will take him more than 10 minutes altogether to wait for the phone and complete his call ?

Answer:

$\frac{1}{\lambda}$ = 12 min; $\mu = \frac{1}{4}$ min,

$$P(\varpi > 10) = e^{-(\mu-\lambda)10} = e\left(\frac{1}{4} - \frac{1}{12}\right)10 = e^{-5/3} = 0.1889$$

23) Customers arrive at a one-man barber shop according to a Poisson process with mean inter-arrival time of 12 minute, Customers spend an average of 10 min in the barber's chair. What is the expected number of customers in the barber shop and in the quene ? How much time can a customer expect to spend in the barber's shop ?

Answer:

$$E(N_S) = \frac{\lambda}{\mu-\lambda}\frac{1/12}{\frac{1}{10}-\frac{1}{12}} = 5 \text{ Customers}$$

$$E(Nq) = \frac{\lambda 2}{\mu(\mu-\lambda)}\frac{1/144}{\frac{1}{10}\left(\frac{1}{10}-\frac{1}{12}\right)} = 4.17 \text{ Customers}$$

$$E[\omega] = \frac{1}{\mu-\lambda}\frac{1}{\frac{1}{10}-\frac{1}{12}} = 60 \text{ min or } 1 \text{ hour}$$

24) A duplication machine maintained for office use is operated by office assistant. The time to complete each job varies according to an exponential distribution with mean 6 min. Assume a Poisson input with an average arrival rate of 5 jobs per hour. If an 8-hour day is used as a base, determine

a) The percentage of idle time of the machine.

b) The average time a job is in the system.

Answer:

$\lambda = 5$ / hour; $\mu = \frac{60}{6}$ 10/ hour

(i) P [the machine is idle] = $P_0 = 1-\frac{\lambda}{\mu} = 1-\frac{5}{10} = \frac{1}{2} = 50\%$

(ii) $E(\omega) = \frac{1}{\mu-\lambda} = \frac{1}{10-5} = \frac{1}{5}$ hours or 12 min

25) In a (M|M|1):(∞/F1F0) queuing model, the arrival and service rates are $\lambda = 12$/ hour and $\mu = 24$/hour, find the average number of customers in the system and in the queue.

Answer:

$$E(N_S) = \frac{\lambda}{\mu-\lambda} = \frac{12}{24-12} = 1 \text{ customer}$$

$$E(N_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{144}{24X12} = \frac{1}{2} \text{ customer}$$

26) Customers arrive at a one-man barber shop according to a Poisson process with a mean inter arrival time of 12 minute. Customers spend an average of 10 minutes in the barber's chair, what is the probability that more than 3 customers are in the system ?

Answer:

$$P[N > 3] = P_4 + P_5 + P_6 + \ldots\ldots$$

$$= 1 - [P_0 + P_1 + P_2 + P_3] = 1 - \left(1-\frac{\lambda}{\mu}\right)\left[1+\left(\frac{\lambda}{\mu}\right)+\left(\frac{\lambda}{\mu}\right)^2+\left(\frac{\lambda}{\mu}\right)^3\right]$$

$$\left(\frac{\lambda}{\mu}\right)^4 = \left(\frac{5}{6}\right)^4 = 0.4823 \qquad \text{Since } \lambda = \frac{1}{2} \ \& \ \mu = \frac{1}{10}$$

27) If a customer has to wait in a (M|M|1):(∞/F1F0) queue system what is his average waiting time in the queue, if λ = 8 per hour and μ=12 per hour ?

Answer:

Average waiting time of a customer in the queue, if he has to wait.

$$= \frac{1}{\mu-\lambda} = \frac{1}{12-8} = \frac{1}{4} \text{ hours (or) 15 min}$$

28) What is the probability that a customer has to wait more than 15 minutes to get his service completed in (M|M|1):(∞/F1F0) queue system, if λ = 6 per hour and μ = 10 per hour.

Answer:

$$\text{Required probability} = e^{-(\mu-\lambda)t} = e^{-(10-6)x\frac{1}{4}=e^{-1}=0.3679}$$

29) If there are two servers in an infinite capacity Poisson queue system with λ=10 and μ=15 per hour, what is the percentage of idle time for each server ?

Answer:

P [the server will be idle] = P_0

$$P_0 = \frac{1}{\sum_{n=0}^{1}\frac{1}{n!}\left(\frac{2}{3}\right)^n + 2!\left(1-\frac{1}{3}\right)^{\left(\frac{2}{3}\right)^2}} = \frac{1}{1+\frac{2}{3}+\frac{1}{3}} = \frac{1}{2}$$

$\therefore$ Percentage of idle time for each server = 50%

30) If λ = 4 per hour and μ = 12 per hour in an (M|M|1):(∞/F1F0) queuing system, find the probability that there is no customer in the system.

Answer:

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1-\frac{1}{3}}{1-\left(\frac{1}{3}\right)^5} = \frac{\frac{2}{3}}{1-\frac{1}{243}} = \frac{2}{3} X \frac{243}{242} = \frac{81}{121}$$

UNIT– V – NON-MARKOVIAN QUEUES AND QUEUE NETWORKS

PART-A

Problem: 1

Write down Pollaczek - Khinchine formula.

Solution:

i) Average number of Customers in the system $=\dfrac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}+\rho$ where $\rho=\dfrac{\lambda}{\mu}$.

ii) Average queue length $=\dfrac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$

Problem: 2

Write the steady-state equations (flow balance equations) for a two-station sequential Queue with blocking.

Solution:

$$\lambda p_{0,0}=\mu_2 p_{0,1}$$

$$\mu_1 p_{1,0}=\mu_2 p_{1,1}+\lambda p_{0,0}$$

$$(\lambda+\mu_2)p_{0,1}=\mu_1 p_{1,0}+\mu_2 p_{b,1}$$

$$(\mu_1+\mu_2)p_{1,1}=\lambda p_{0,1}$$

$$\mu_2 p_{b,1}=\mu_1 p_{1,1}$$

Problem: 3

Write the flow-balance equations of open Jackson networks.

Solution:

$$\lambda_j=r_j+\sum_{i=1}^{k}\lambda_i P_{ij} \qquad j=0,1,2,...k$$

where P_{ij} is the probability that a departure from server i joins the Queue at server j.

Problem: 4

Write the flow-balance equations of closed Jackson networks.

Solution:

$$\lambda_j=\sum_{i=1}^{k}\lambda_i P_{ij} \qquad j=0,1,2,...k$$

where P_{ij} is the probability that a departure from server i joins the Queue at server j.

Problem: 5

State the characteristics of Jackson networks.

Solution:

(a) Arrivals from outside through node i follow a Poisson process with mean arrival rate r_i.

(b) Service times at each channel at node i are independent and exponentially distributed with parameter μ_i

(c) The probability that a customer who has completed service at node i will go next to node j (Routing probability) is P_{ij} , i= 1, 2, 3, ….k and j = 0, 1, 2, 3, …k. and P_{i0} denotes the probability that a customer will leave the system from node i. P_{ij} is independent of the state of the system.

If λ_j is the total arrival rate of customers to node j, $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij} \quad {}_{j=0,1,2,\ldots k}$

Problem: 6

Distinguish between open Jackson networks and closed Jackson networks

Solution: We have,

$$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij} \quad {}_{j=0,1,2,\ldots k}$$

In all general cases where, $r_i \neq 0$ for any i or $P_{i0} \neq 0$for any i, the Jackson networks are referred to as open Jackson networks. In the case of closed Jackson networks, $r_i = 0$ for all i and $P_{i0} = 0$for all i.

PART-B

Problem: 7

Derive Pollaczek - Khinchine relation for $(M/G/1):(\infty/GD)$

Solution:

In order to determine the mean queue length L_q and mean waiting time Wq for this system the following technique is used.

Let $f(t)$ = Probability distribution of service time t with mean $E(t)$ and Variance $V(t)$.

$n =$ no. of customers in the system just after a customer departs.

$t =$ service time of the customer following the one already departed.

$n^1 =$ The number of customers left behind the next departing customer.

$n^1 = k \quad if \quad n = 0$

$= (n-1) + k \quad if \quad n > 0$

Where k = 0,1,2….is the number of arrivals during the service time.

Alternatively of $\delta = 1 \quad if \quad n = 0$

$= 0 \quad if \quad n > 0$

$n^1 = n - 1 + \delta + k \quad - \quad (1)$

Then taking expectation

$E\left(n^{1}\right)=E(n)+E(\delta)+E(k)-1$ Since $E\left(n^{1}\right)=E(n)$ in steady state $E(\delta)=1-E(k)$

Squaring (1) both sides

$$\left(n^{1}\right)^{2}=\left[n+(k-1)+\delta\right]^{2}=n^{2}+(k-1)^{2}+2n(k-1)+\delta^{2}+2n\delta+2(k-1)\delta.$$

$$=n^{2}+(k-1)^{2}+2n(k-1)+\delta^{2}+2n\delta+2k\delta-2\delta$$

Since δ can take values 0 or 1only $\delta^{2}=\delta \quad and \quad n\delta=0$

$$\therefore n^{1^{2}}=n^{2}+k^{2}-2k+1+2n(k-1)+\delta+2k\delta-2\delta$$

$$=n^{2}+k^{2}+2n(k+1)+\delta(2k-1)-2k+1$$

$$2n(1-k)=n^{2}-n^{1^{2}}+k^{2}+\delta(2k-1)-2k+1$$

Taking expectation on both sides

$$2E(n)\left[1-E(k)\right]=E\left(n^{2}\right)-E\left(n^{1^{2}}\right)+E\left(k^{2}\right)+E(\delta)(2E(k)-1)-2E(k)+1$$

$$E(n)=\frac{E\left(k^{2}\right)-2E(k)+E(\delta)\left(2E(k)-1\right)+1}{2\left(1-E(k)\right)}$$

$$E(\delta)=1-E(k)$$

$$E(n)=\frac{E\left(k^{2}\right)-2E(k)+\left[1-E(k)\right]\left(2E(k)-1\right)+1}{2\left(1-E(k)\right)}$$

$$.\,E(n)=\frac{E\left(k^{2}\right)+E(k)-2E^{2}(k)}{2\left(1-E(k)\right)}$$

Now is order to determine $E(n)$, the values of $E(k)$ and $E\left(k^{2}\right)$ are to be computed. Since the arrivals follow Poisson distribution $E(k)=\int_{0}^{\infty}E(k/t)f(t)\,dt$

Now $E(k/t)=\lambda t, E\left(k^{2}/t\right)=(\lambda t)^{2}+(\lambda t)$

$$V(k/t)=E\left(k^{2}/t\right)-E^{2}(k/t) \qquad \therefore E(k)=\int_{0}^{\infty}\lambda t\, f(t)\,dt=\lambda E(t)$$

$$E\left(k^{2}\right)=\int_{0}^{\infty}E\left(k^{2}/t\right)f(t)\,dt=\int_{0}^{\infty}\left[(\lambda t)^{2}+\lambda t\right]f(t)\,dt$$

$$=\lambda^{2}E\left(t^{2}\right)+\lambda E(t)=\lambda^{2}\left[V(t)+\left[E^{2}(t)\right]\right]+\lambda E(t)$$

$$E\left(k^{2}\right)=\lambda^{2}\left[V(t)+E^{2}(t)\right]+\lambda E(t)$$

$$\therefore E(n)=\frac{\lambda^{2}\left[V(t)+E^{2}(t)\right]+\lambda E(t)+\lambda E(t)-2\lambda^{2}E^{2}(t)}{2\left[1-\lambda E(t)\right]}$$

$$= \frac{\lambda^2 V(t) + \lambda^2 E^2(t) + 2\lambda E(t) - 2\lambda^2 E^2(t)}{2(1-\lambda E(t))} = \frac{\lambda^2 V(t) + \lambda^2 E^2(t) + 2\lambda E(t)\left[1-\lambda E(t)\right]}{2(1-\lambda E(t))}$$

$$= \lambda E(t) + \frac{\lambda^2 E^2(t) + \lambda^2 V(t)}{2(1-\lambda E(t))}$$

$$L_s = E(n) = \lambda E(t) + \frac{\lambda^2\left[E^2(t) + V(t)\right]}{2(1-\lambda E(t))} \qquad W_s = \frac{L_s}{\lambda}$$

$E(t) = 1/\mu$ where μ is the rate of servive.

Then $L_s = \lambda/\mu + \dfrac{\lambda^2\left[(1/\mu)^2 + \sigma^2\right]}{2(1-\lambda/\mu)} = \rho + \dfrac{(\rho^2 + \lambda^2\sigma^2)}{2(1-\rho)}$ where $\lambda/\mu = \rho$ and $V(t) = \sigma^2$

$$L_q = L_s - \rho = \frac{(\rho^2 + \lambda^2\sigma^2)}{2(1-\rho)}$$

Problem: 8

A one man barber shop takes exactly 25 minutes to complete one haircut. If customers arrive at the barber shop in a Poisson fashion at an average rate of one every 40 minutes, how long on the average a customer spends in the shop? Also find the average time a customer must wait for service.

Solution:

Since the service time T is a constant=25 minutes, T follows a probability distribution with E(T)=25, Var(T)=0.

$\lambda = 1$ for every 40 minutes

$= \dfrac{1}{40}$ per minutes

By Pollaczek-Khinchine formula,

$$L_s = E(T) + \frac{\lambda^2\left[\operatorname{var}(T) + \left(E(T)\right)^2\right]}{2(1-\lambda E(T))}$$

$$= \frac{25}{40} + \frac{\left(\frac{1}{40}\right)^2\left[0+25^2\right]}{2\left(1-\frac{1}{40}.25\right)}$$

$$= \frac{25}{40} + \frac{\frac{1}{1600}.625}{2\left(\frac{15}{40}\right)} = \frac{25}{40} + \frac{625}{1600} \times \frac{4}{3} = \frac{5}{8} + \frac{25}{48} = \frac{55}{48}$$

By Little formula,

$$W_S = \frac{L_s}{\lambda} = \frac{\frac{55}{48}}{\frac{1}{40}} = \frac{55}{48} \times 40$$

$= 45.8$ Minutes

$$W_q = W_s - \frac{1}{\mu}$$

$$= 45.8 - E(T) \qquad \left[\therefore \mu = \frac{1}{E(T)}\right]$$

$= 45.8 - 25$

$= 20.8$ Minutes

Hence, a customer has to spend 45.8 minutes in the shop and has to wait for service for 20.8 minutes on the average.

Problem: 9 Suppose a one person tailor shop is in business of making men's suits. Each suit requires four district tasks to be performed before it is completed. Assume all four tasks must be completed on each suit before another is started. The time to perform each task has an exponential distribution with a mean of 2 hr. If orders for a suit come at the average rate 5.5 per week (assume an 8hr day, 6 day week), how long can a customer expect to wait to have a suit made?

Solution: Given: $\lambda = 5.5$ per week

$$= \frac{5.5}{6 \times 8} \text{ orders/hr}$$

=0.1149 orders/hr

The server time for each task $= \dfrac{1}{k\mu}$ where $k = 4$

i.e., $= \dfrac{1}{4\mu} = 2\,hr.$ (given)

$$\mu = \frac{1}{8} hr$$

=0.125 orders / hr.

$$\rho = \frac{\lambda}{\mu} = \frac{0.1149}{0.125} = 0.9192$$

Since the service time T is constant then $\sigma^2 = 0$

Here the average service rate μ is the average service rate to complete a suit.

The expected waiting time of a customer (in the system) is

$$W_s = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} + \frac{1}{\mu}$$

Problem: 10 In a heavy machine shop, the overhead crane is 75% utilized. Time study observations gave the average slinging time as 10.5 minutes with a standard deviation of 8.8 minutes. What is the average calling rate for the services of the crane and what is the average delay in getting service? If the average service time is cut to 8.0 minutes, with a standard deviation of 6.0 minutes, how much reduction will occur, on average, in the delay of getting served?

Solution: This is a $(M/G/1):(\infty/FIFO)$ Process

Given: Utilization rate $= 75\% = \dfrac{75}{100} = \dfrac{3}{4}$

i.e., $\rho = \dfrac{3}{4}$

Average service time E(T)=10.5 min

$$\mu = \frac{1}{E(T)} = \frac{1}{10.5}$$

We know that $\rho = \dfrac{\lambda}{\mu}$

$\Rightarrow \lambda = \rho\mu$

$$\lambda = \left(\frac{3}{4}\right)\left(\frac{1}{10.5}\right) = 0.0714\,\text{min}$$

i.e., average calling rate for the services of the crane

$\lambda = 0.0714\,\text{min} = 4.286$ hour

To find the average delay in getting service

Here $\sigma = 8.8, \lambda = 0.0714, \rho = \dfrac{3}{4} = 0.75$

$\therefore$ Average delay in getting service

$$W_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} = \frac{(0.0714)^2(8.8)^2 + (0.75)^2}{2(0.0714)(1-0.75)} = 26.815 \text{ min}$$

i.e., The average service time is cut to 8.3 minutes then

$$\mu = \frac{1}{E(T)} = \frac{1}{8.0}\text{min}$$

$$\lambda = 0.0714\,\text{min}$$

$$\rho = \frac{\lambda}{\mu} = \frac{0.0714}{(1/8)} = 0.5712$$

$$\sigma = 6\,\text{min}$$

$\therefore$ Average delay in getting service

$$W_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} = \frac{(0.0714)^2(6)^2 + (0.5712)^2}{2(0.0714)(1-0.5712)}$$

$$= 8.326 \text{ min}$$

$\therefore$ The Average waiting time has a reduction of
26.8-8.32=18.5 minutes

$$= \frac{0 + (0.9192)^2}{2(0.1149)(0.0808)} + \frac{1}{0.125} = 53.42 \text{ hrs} = 1.113 \text{ Weeks}.$$

Problem: 11 A car manufacturing plant uses one big crane for loading cars into a truck. Cars arrive for loading by the crane according to a Poisson distribution with a mean of 5 cars per hour. Given that the service time for all cars is constant and equal to 6 minutes determine $L_S, L_q, W_S \quad and \quad W_q$.

Solution: The given problem is in $(M/G/1):(\infty/FIFO)$ model.

$\lambda = 5\,\text{cars/hr}$ cars/hr.

The service time T is constant with mean $E[T] = \frac{1}{\mu} = 6$ min

Since the service time T is constant then var[T]= $\sigma^2 = 0$

$$\frac{1}{\mu} = 6 \text{ min} = 6 \times \frac{1}{60} hr = \frac{1}{10} hr.$$

$$\mu = 10 \; hr$$

$$\rho = \frac{\lambda}{\mu} = \frac{5}{10} = \frac{1}{2} hr$$

Using $\rho - k$ formula, we have

$$L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

$$= \frac{1}{2} + \frac{0 + (1/4)}{2(1-1/2)} = \frac{3}{4} = 0.75 \;\; car/hr$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= \frac{3}{4} - \frac{1}{2}$$

$$= 0.25 \;\; car/hr$$

$$W_s = \frac{L_s}{\lambda} = \frac{(3/4)}{(1/5)} = \frac{3}{20} hr = 9 \text{ minutes}$$

$$W_q = \frac{L_q}{\lambda} = \frac{(1/4)}{(1/5)} = \frac{1}{20} hr = 3 \text{ minutes.}$$

Problem: 12 A car wish facility operates with only one bay. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. The parking lot is large enough to accommodate any number of cars. Find the average number of cars waiting in the packing lot, if the time for washing and cleaning a car follows.

(a) Uniform distribution between 8 and 12 minutes

(b) A normal distribution with mean 12 minutes and S.D.3minutes.

(c) A discrete distribution with values equal to 4,8 and 15 minutes and corresponding probabilities 0.2,0.6 and 0.2.

Solution:

$$\text{Mean} = \lambda = 4/\text{hour} = \frac{4}{60} \text{ per minute}$$

$$= \frac{1}{15} \text{ per minute}$$

Let T be a continuous random variable.

Then, E(T) = mean of the uniform distribution $= \frac{1}{2}(a+b) = \frac{1}{2}(8+12) = \frac{1}{2}(20) = 10$

$$\text{Var (T)} = \frac{1}{12}(b-a)^2$$

$$= \frac{1}{12}(12-8)^2 = \frac{1}{12}(4)^2 = \frac{1}{12}(16) = \frac{4}{3}$$

By the Pollaczek-Kninchine formula,

$$L_s = \lambda.E(T) + \frac{\lambda^2\left[\text{var}(T) + (E(T))^2\right]}{2(1-\lambda E(T))}$$

$$= \frac{1}{15}.10 + \frac{\left(\frac{1}{15}\right)^2\left[\frac{4}{3} + (10)^2\right]}{2\left(1 - \frac{1}{15}.10\right)}$$

$$= \frac{2}{3} + \frac{\frac{1}{225}\left[\frac{4}{3} + 100\right]}{2\left(1 - \frac{2}{3}\right)}$$

$$= \frac{2}{3} + \frac{\frac{1}{225}.\frac{304}{3}}{2\left(\frac{1}{3}\right)} = \frac{2}{3} + \frac{1}{225}.\frac{304}{3}.\frac{3}{2} = \frac{2}{3} + \frac{152}{225} = \frac{302}{225}$$

$$= 1.342 \text{ Cars}$$
$$\cong \text{ Car}$$

By Little's formula,

$$L_q = L_s - \frac{\lambda}{\mu}$$
$$= 1.342 - \frac{\frac{1}{15}}{\frac{1}{10}}$$
$$= 0.675 \text{ cars}$$
$$\cong 1 \text{ car}$$

(b) $\lambda = \frac{1}{15}$, $E(T) = 12$ min, and Var(T)=9

$$\therefore \qquad \mu = \frac{1}{E(T)} = \frac{1}{12}$$

By the Pollaczek-Kninchine formula,

$$L_s = \lambda.E(T) + \frac{\lambda^2\left[\operatorname{var}(T) + (E(T))^2\right]}{2(1 - \lambda E(T))}$$

$$= \frac{1}{15}.12 + \frac{\left(\frac{1}{15}\right)^2\left[9 + 12^2\right]}{2\left(1 - \frac{1}{15}.12\right)} = \frac{4}{5} + \frac{\frac{1}{225}.153}{2\left(\frac{3}{15}\right)} = \frac{4}{5} + \frac{153}{225}.\frac{15}{6}$$

$$= \frac{4}{5} + \frac{153}{90} = 2.5 \text{ Cars.}$$

By Little's formula,

$$L_q = L_s - \frac{\lambda}{\mu}$$
$$= 2.5 - \frac{\frac{1}{15}}{\frac{1}{12}} = 2.5 - \frac{12}{15} = 2.5 - 0.8 = 1.7\, Cars \approx 2\, Cars$$

(c) The service time T follows the discrete distribution given below.

T :	4	8	15
P(T) :	0.2	0.6	0.2

$$\mathrm{E(T)} = \sum TP(T)$$
$$= 4 \times 0.2 + 8 \times 0.6 + 15 \times (0.2)$$
$$= 0.8 + 4.8 + 3$$
$$= 8.6 \text{ minutes}$$

$$\mathrm{E}\left(\mathrm{T}^2\right) = \sum T^2 P(T)$$

$$= 4^2(0.2) + 8 \times 0.6 + 15 \times (0.2) = 0.8 + 4.8 + 3 = 8.6\text{minutes}$$

$$\mathrm{var}(\mathrm{T}) = E\left(T^2\right) - \left[E(T)\right]^2$$

$$= 86.6 - (8.6)^2 = 12.64$$

BY Pollaczek-Khinchine formula,

$$L_s = \lambda.E(T) + \frac{\lambda^2\left[\mathrm{var}(T) + \left(E(T)\right)^2\right]}{2\left(1 - \lambda\, E(T)\right)}$$

$$= \frac{1}{15} \times 8.6 + \frac{\frac{1}{225}\left[12.64 + (8.6)^2\right]}{2\left[1 - \frac{1}{15} \times 8.6\right]}$$

$$= 0.573 + \frac{\frac{1}{225}\left[12.64 + 73.96\right]}{2\left[1 - \frac{8.6}{15}\right]} = 0.573 + \frac{86.6}{225} \times \frac{15}{12.8} = 1.024 \approx 1\,Car$$

By Little formula,

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= 1.024 - \frac{\frac{1}{5}}{\frac{1}{86}} \qquad \left[\therefore\ \mu = \frac{1}{E(T)}\right]$$

$$= 1.025 - \frac{8.6}{5} = 0.45\ Car$$

Problem: 13 An automatic car wash facility operates with only one bay; Cars arrive according to a Poisson process with mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. If the service times for all cars is constant and equal to 10 minutes. Determine L_q, L_q, W_s, W_q.

Solution:

$$\lambda = 4\,\text{per hour}$$

$$= \frac{4}{60}\ \text{per minute}$$

$$= \frac{1}{15}\text{per minute}$$

Given service time 'T' is a constant.

$\therefore$ 'T' follows a distribution with E(T)=10 and Var(T)=0.

By Pollaczek - Khinchine formula,

$$L_s = \lambda.E(T) + \frac{\lambda^2\left[\operatorname{var}(T) + (E(T))^2\right]}{2(1-\lambda E(T))}$$

$$= \frac{1}{15}\times 10 + \frac{\left(\frac{1}{15}\right)^2\left[0+10^2\right]}{2\left(1-\frac{1}{15}.10\right)}$$

$$= \frac{10}{15} + \frac{\frac{100}{225}}{2\left(\frac{15-10}{15}\right)} = \frac{10}{15} + \frac{100}{225}\times\frac{15}{10} = \frac{20}{15} = \frac{4}{3}$$

By Little formula,

$$W_S = \frac{L_s}{\lambda} = 15\times\frac{4}{3} = 20 \text{ minutes}$$

$$W_q = W_s - \frac{1}{\mu}$$

$$= 20 - E(T) \qquad \left[\therefore\ \mu = \frac{1}{E(T)}\right]$$

$$= 20-10$$

$= 10$ minutes.

i.e., a customer has to spend 20 minutes in the system and 10 minutes in the queue.

$$L_q = \lambda.W_q$$

$$= \frac{1}{15}\times 10 = \frac{10}{15}$$

Problem: 14 In a big factory, there are a large number of operating machines and two sequential repair shops, which do the service of the damaged machines exponentially with respective rates of 1/hour and 2/hour. If the cumulative failure rate of all the machines in the factory is 0.5 / hour, find (i) the probability that both repair shops are idle, (ii) the average number of machines in the service section of the factory and (ii) The average repair time of a machine.

Solution: Given $\lambda = 0.5$ / hour

$= \frac{1}{2}$ per hour

$\mu_1 = 1$ per hour

$\mu_2 = 2$ per hour

The situation in this problem is comparable with 2-stage Tandem queue with single server at each state.

(i) P(both the service stations are idle)

$$= P(0,0)$$

$$= \left(\frac{\lambda}{\mu_1}\right)^0 \cdot \left(1-\frac{\lambda}{\mu_1}\right) \cdot \left(\frac{\lambda}{\mu_2}\right)^0 \cdot \left(1-\frac{\lambda}{\mu_2}\right)$$

$$= \left(\frac{1/2}{1}\right)^0 \left(1-\frac{1/2}{1}\right)\left(\frac{1/2}{1}\right)^0 \left(1-\frac{1/2}{1}\right)$$

$$= \left(\frac{1}{2}\right)\left(1-\frac{1}{4}\right) = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8}$$

(ii) The average number of machines in service

$$= \frac{\lambda}{\mu_1-\lambda} + \frac{\lambda}{\mu_2-\lambda}$$

$$= \frac{1/2}{1-\frac{1}{2}} + \frac{\frac{1}{2}}{2-\frac{1}{2}} = \frac{1/2}{1/2} + \frac{1/2}{3/2} = 1 + \frac{1}{3} = \frac{4}{3}$$

(iii) The average repair time

$$= \frac{1}{\mu_1-\lambda} + \frac{1}{\mu_2-\lambda}$$

$$= \frac{1}{1-\frac{1}{2}} + \frac{1}{2-\frac{1}{2}} = \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{\left(\frac{3}{2}\right)} = \frac{8}{3}$$

Problem: 15 A TVS company in Chennai containing a repair section shared by a large number of machines has 2 sequential stations with respective service rates of 3 per hour and 4 per hour. The cumulative failure rate of all the machines is 1 per hour. Assuming that the system behavior can be approximated by the above 2-stage tendon queue, find

(i) the probability that booth the service stations are idle (free)
(ii) the average repair time including the waiting time.
(iii) the bottleneck of the repair facility.

Solution:

STEP-1: Model Identification

The current situation comes under the sequence queue model, Since any number of machines can be repaired, each station comes under the model $(M/M/1):(\infty/FCFS)$

STEP 2: Given Data

Cumulative failure rate $\lambda = 1$

Service rate of station I $\mu_1 = 3$

Service rate of station II $\mu_2 = 4$

STEP 3: To find the following

(1) The probability that both the servive stations are idle.
(2) The average repair time and waiting time
(3) The bottleneck of the repair facility.

STEP 4: Required Computations

(i) P(m customers in the I station and n customers in the II station

$$P_{mn} = \left(\frac{\lambda_1}{\mu_1}\right)^m \left(1 - \frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_2}{\mu_2}\right)^n \left(1 - \frac{\lambda_2}{\mu_2}\right)$$

$$P_{00} = \left(\frac{1}{3}\right)^0 \left(1 - \frac{1}{3}\right)\left(\frac{1}{4}\right)^0 \left(1 - \frac{1}{4}\right) = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$$

(ii) The average number of machines in service at the system(both the stations)

$$L_s = \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda}$$

$$= \frac{1}{3-1} + \frac{1}{4-1} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

= 1 machine (approximately)

Average repair time (including waiting time)

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}$$

$= \frac{5}{6}$ hours =50 minutes.

(iii) Since $\frac{\lambda}{\mu_1} = \frac{1}{3} > \frac{\lambda}{\mu_2} = \frac{1}{4}$

We get the service station 1 is the bottleneck of the repair facility.

Problem: 16 In the Airport reservation section of a city junction, there is enough space for the customers to assemble, form a queue and fill up the reservation forms. There are 5 reservation counters in front of which also there is enough space for the customers to wait. Customers arrive at the reservation counter section at the rate of 40 per hour and takes one minute on the average to fill up the forms. Each reservation clerk takes 5 minutes on the average to complete the business of a customer in an exponential manner.(i) Find the probability that a customer has to wait to get the service in the reservation counter section (ii) Find the total waiting time for a customer in the entire reservation section. Assume that only those who have the filled up reservation forms will be allowed into the counter section.

Solution:

The queuing system in the form-filling portion is only a M/M/1 model, since each customers is served by himself or herself (viz., one server)

For this system, $\lambda = 40/$ hour and $\mu = 60$ /hour.

$$E(N_s) = \frac{\lambda}{\mu - \lambda} = \frac{40}{60-40} = \frac{40}{20} = 2$$

$$E(W_s) = \frac{1}{\mu - \lambda} = \frac{1}{20} \text{ hour or 3 min}$$

The queuing system in the reservation counter section is an M/M/s model with $\lambda = 40$/hour, $\mu = 12$/hour and $s = 5$ [Since the output of the M/M/1 system in the same as the input of that system, by Burke's theorem and the output of this system (namely, 40/hour) becomes the input of the M/M/s system]

(i) P(a customer has to wait in the counter section)

$$= \frac{\left(\frac{\lambda}{\mu}\right)^2 .P_0}{\lfloor s \left(1 - \frac{\lambda}{\mu s}\right)}, \quad ....(1)$$

$$P_0 = \left[\sum_{r=0}^{s-1} \frac{1}{\lfloor r} . \left(\frac{\lambda}{\mu}\right)^r + \left\{\frac{(\lambda/\mu)^s}{\lfloor s \left(1 - \frac{\lambda}{\mu s}\right)}\right\}\right]^{-1}$$

Where $\left[\sum_{r=0}^{4-1} \frac{1}{\lfloor r} . \left(\frac{10}{3}\right)^r + \frac{\left(\frac{10}{3}\right)^5}{5!(1-2/3)}\right]^{-1}$

$$= \frac{1}{0!}\left(\frac{10}{3}\right)^{10} + \frac{1}{\lfloor 1}\left(\frac{10}{3}\right)^1 + \frac{1}{\lfloor 2}\left(\frac{10}{3}\right)^2 + \frac{1}{\lfloor 3}\left(\frac{10}{3}\right)^3 + \frac{1}{\lfloor 4}\left(\frac{10}{3}\right)^4 + \frac{\frac{100000}{243}}{(120)\left(\frac{1}{3}\right)}\Bigg]^{-1}$$

$$= \left[1 + \frac{10}{3} + \frac{100}{18} + \frac{1000}{162} + \frac{1}{24} + \frac{100000}{81} + \left(\frac{100000}{243}\right)\left(\frac{3}{120}\right)\right]^{-1}$$

$$= \left[1 + \frac{10}{3} + \frac{100}{18} + \frac{1000}{162} + \frac{1}{24}\frac{10000}{81} + 10.288\right]^{-1}$$

$$= [21.206 + 10.288]^{-1}$$

$$= [31.494]^{-1} = 0.032$$

$$(1) \Rightarrow \therefore \text{ Required probability } = \frac{\left(\frac{10}{3}\right)^5 (0.032)}{5!\left(1-\frac{2}{3}\right)}$$

$$= \frac{\left(\frac{100000}{243}\right)(0.032)}{(120)\left(\frac{1}{3}\right)} = 0.3292$$

$$E[N_s] = \frac{1}{ss!}\frac{(\lambda/\mu)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2}P_0 + \frac{\lambda}{\mu}$$

$$= \frac{1}{(5)(5!)}\frac{\left(\frac{10}{3}\right)^6}{\left(1-\frac{2}{3}\right)}(0.032) + \frac{10}{3}$$

$$= \frac{1}{(5)(120)}\left[\frac{\left(\frac{1000000}{729}\right)}{\left(\frac{1}{3}\right)}(0.032)\right] + \frac{10}{3}$$

$$= \frac{1}{600}[131.69] + 3.3333 = 0.2195 + 3.3333 = 3.5528$$

$$\therefore E(W_s) = \frac{1}{\lambda}E[N_s] = \frac{1}{40}(3.5528)$$

$= 0.0888$ *hour*

$= 5.329$ minutes

$\therefore$ Total waiting time of a customer in the entire reservation room =6+5.329=11.3292 minutes.

Problem: 17 The last two things that are done to a car before its manufacture is complete are installing the engine and putting on the tires. An average of 54 cars per hour arrives, requiring these two tasks. One worker is available to install the engine and can service an average of 60 cars per hour. After the engine is installed, the car goes to the tire station and waits for its tires to be attached. Three workers serve at the tire station. Each works on one car at a time and can put tires on a car in an average of 3 minutes. Both inter arrival times and service times are exponential.

(i) Determine the mean queue length at each work station.

(ii) Determine the total expected time a car spends waiting for service.

Solution: $\lambda = 54, s_1 = 1, \mu_1 = 60, s_2 = 3, \mu_2 = 20$

Since $\lambda < \mu_1$ *and* $\lambda < 3\mu_2$, neither queue will "below up" and Jackson's theorem is applicable.

For stage 1 (engine)

$$\rho = \frac{\lambda}{s_1\,\mu_1} = \frac{54}{60} = 0.90$$

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.90)^2}{1-0.90} = \frac{0.81}{0.1} = 8.1\,cars$$

$$W_q = \frac{1}{\lambda}L_q = \frac{1}{54}(8.1) = 0.15\,hour$$

For stage 2 (tires)

$$\rho = \frac{\lambda}{s_2\,\mu_2} = \frac{54}{3(20)} = \frac{54}{60} = 0.90$$

$$\rho(j \ge 3) = 0.83$$

$$L_q = \frac{(0.83)(0.90)}{1-0.90} = 7.47\,cars$$

$$W_q = \frac{1}{\lambda}L_q = \frac{1}{54}(7.47) = 0.138\,hour$$

Thus, the total expected time a car spends waiting for engine installation an tires is 0.15+0.138=0.288 hour.

Problem: 18 For a 2- stage (service point) sequential queue model with blockage, compute the average number of customers in system and the average time that a customer has to spend in the system, if $\lambda = 1$, $\mu_1 = 2$ *and* $\mu_2 = 1$

Solution:

Given $\lambda = 1, \mu_1 = 2, \mu_2 = 1$.

The balanced equations are

State	Rate that the process leaves	
(0,0)	$\lambda P_{00} = \mu_2 P_{01}$	(1)
(1,0)	$\mu_1 P_{00} = \lambda P_{00} + \mu_2 P_{11}$	(2)
(0,1)	$(\lambda + \mu_2) P_{01} = \mu_1 P_{10} + \mu_1 P_{b1}$	(3)
(1,1)	$(\mu_1 + \mu_2) P_{11} = \lambda P_{01}$	(4)
(b,1)	$\mu_2 P_{b1} = \mu_1 P_{11}$	(5)
	$P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$	(6)
(1)	$\Rightarrow P_{00} = P_{01}$	(7)
(2)	$\Rightarrow 2P_{10} = P_{00} + P_{11}$	(8)
(3)	$\Rightarrow 2P_{01} = 2P_{10} + P_{b1}$	(9)
(4)	$\Rightarrow 3P_{11} = P_{01}$	(10)
(5)	$\Rightarrow Pb_1 = 2P_{11}$	(11)

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$$(7) \ \& \ (10) \qquad \Rightarrow P_{00} = P_{01} = 3P_{11} \qquad (12)$$

$$(12) \ \& \ (1) \qquad \Rightarrow P_{00} = P_{01} = 3P_{11} = \frac{3}{2}P_{b_1} \qquad (13)$$

$$(8) \ \Rightarrow 2P_{10} = P_{00} + \frac{1}{3}P_{00}$$

$$2P_{10} = \frac{4}{3}P_{00}$$

$$P_{10} = \frac{2}{3}P_{00} \qquad (14)$$

$$(6) \ \Rightarrow P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$$

$$P_{00} + \frac{2}{3}P_{00} + P_{00} + \frac{1}{3}P_{00} + \frac{2}{3}P_{00} = 1 \qquad \text{by (13) \& (14)}$$

$$P_{00}\left[1 + \frac{2}{3} + 1 + \frac{1}{3} + \frac{2}{3}\right] = 1$$

$$P_{00}\left[2 + \frac{5}{3}\right] = 1$$

$$P_{00}\left[\frac{11}{3}\right] = 1$$

$$\boxed{P_{00} = \frac{3}{11}}$$

$$(13) \ \Rightarrow P_{01} = \frac{3}{11}$$

$$P_{11} = \frac{1}{3}P_{00} = \frac{1}{3}\left(\frac{3}{11}\right) = \frac{1}{11}$$

$$\boxed{P_{11} = \frac{1}{11}}$$

i.e.,

$$(14) \ \Rightarrow P_{10} = \frac{2}{3}\left(\frac{3}{11}\right) = \frac{2}{11} \qquad \textit{i.e.,} \quad \boxed{P_{10} = \frac{2}{11}}$$

$$(13) \ \Rightarrow P_{b1} = \frac{2}{3}P_{00} \quad = \frac{2}{3}\left(\frac{3}{11}\right) \qquad P_{b1} = \frac{2}{11}$$

$$L = P_{01} + P_{10} + 2\left(P_{11} + P_{b1}\right)$$

$$= \frac{3}{11} + \frac{2}{11} + 2\left(\frac{1}{11} + \frac{2}{11}\right)$$

$$= \frac{5}{11} + 2\left(\frac{3}{11}\right) = \frac{5}{11} + \frac{6}{11} = 1$$

$$W = \frac{P_{01} + P_{10} + 2\left(P_{11} + P_{b1}\right)}{\lambda\left(P_{00} + P_{01}\right)} = \frac{L}{\lambda\left(P_{00} + P_{01}\right)}$$
$$= \frac{1}{1\left(\frac{3}{11} + \frac{3}{11}\right)} = \frac{1}{(6/11)} = \frac{11}{6}$$

Problem: 19 For a 2-stage (service point) sequential queue model with blockage, compute L_s and $W_S, if\ \lambda = 1, \mu_1 = 1 and\ \mu_2 = 2$.

Solution: Given: $\lambda = 1, \mu_1 = 1, \mu_2 = 2$

The balanced equations are

$(0,0)$ $\quad \lambda P_{00} = \mu_2 P_{01}$ .....(1)

$(1,0)$ $\quad \mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$ .....(2)

$(0,1)$ $\quad \left(\lambda + \mu_2\right) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$ .....(3)

$(1,1)$ $\quad \left(\mu_1 + \mu_2\right) P_{11} = \lambda P_{01}$ .....(4)

$(b,1)$ $\quad \mu_2 P_{b1} = \mu_1 P_{11}$ .....(5)

$P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$ .....(6)

$(1) \Rightarrow P_{00} = 2P_{01}$ .....(7)

$P_{10} = P_{00} + 2P_{11}$ .....(8)

$3P_{01} = P_{10} + 2Pb_1$ .....(9)

$3P_{11} = P_{01}$ .....(10)

$2P_{b1} = P_{11}$ .....(11)

$(7)\&(10) \Rightarrow P_{00} = 2P_{01} = 6P_{11}$ .....(12)

$(12)\&(11) \Rightarrow P_{00} = 2P_{01} = 6P_{11} = 12P_{b1}$ .....(13)

$(8) \Rightarrow P_{10} = P_{00} + \frac{1}{3}P_{00}$

$P_{10} = \frac{4}{3}P_{00}$ .....(14)

$$(6) \Rightarrow P_{00} + \frac{4}{3}P_{00} + \frac{1}{2}P_{00} + \frac{1}{6}P_{11} + \frac{1}{12}P_{00} = 1\ by\ (13)\&(14)$$

$$P_{00}\left[1 + \frac{4}{3} + \frac{1}{2} + \frac{1}{6} + \frac{1}{12}\right] = 1$$

$$P_{00}\left[\frac{37}{12}\right] = 1$$

$$\boxed{P_{00} = \frac{12}{37}}$$

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$$(13) \qquad P_{01} = \frac{1}{2} P_{00}$$

$$= \frac{1}{2}\left(\frac{12}{37}\right)$$

$$\boxed{P_{01} = \frac{6}{37}}$$

$$(13) \qquad P_{11} = \frac{1}{6} P_{00}$$

$$= \frac{1}{6}\left(\frac{12}{37}\right)$$

$$\boxed{P_{11} = \frac{2}{37}}$$

$$(13) \qquad P_{b1} = \frac{1}{12} P_{00}$$

$$= \frac{1}{12}\left(\frac{12}{37}\right)$$

$$\boxed{P_{b1} = \frac{1}{37}}$$

$$(14) \qquad P_{10} = \frac{4}{3} P_{00}$$

$$= \frac{4}{3}\left(\frac{12}{37}\right)$$

$$\boxed{P_{10} = \frac{16}{37}}$$

$$\therefore L = P_{01} + P_{10} + 2\left(P_{11} + P_{b1}\right)$$

$$= \frac{6}{37} + \frac{6}{37} + 2\left(\frac{2}{37} + \frac{1}{37}\right) = \frac{22}{37} + 2\left(\frac{3}{37}\right) = \frac{28}{37}$$

$$W = \frac{L}{\lambda\left(P_{00} + P_{01}\right)} = \frac{(28/37)}{1\left(\frac{12}{37} + \frac{6}{37}\right)} = \frac{(28/37)}{(18/37)} = \frac{14}{9}$$

Problem: 20 There are two salesmen in a shop, one in charge of receiving payment and the other in charge of delivering the items. Due to limited availability of space, only one customer is allowed to enter the shop, that too when the clerk is free. The customer who has finished his job has to wait there until the delivery section becomes free. It customers arrive in accordance with a Poisson process at rate 1 and the service times to two clerks are independent and have exponential rates of 1 and 3, find

(i) the proportion of customers who enter the ration shop
(ii) the average number of customers in the shop and
(iii) the average amount of time that an entering customer spends in the shop.

Solution: Given : $\lambda = 1, \mu_1 = 1, \mu_2 = 3$

The balanced equations are

$(0,0) \qquad \lambda P_{00} = \mu_2 P_{01} \qquad .....(1)$

$(1,0) \qquad \mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11} \qquad .....(2)$

$(0,1) \qquad (\lambda + \mu_2) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1} \qquad .....(3)$

$(1,1) \qquad (\mu_1 + \mu_2) P_{11} = \lambda P_{01} \qquad .....(4)$

$(b,1) \qquad \mu_2 P_{b1} = \mu_1 P_{11} \qquad .....(5)$

$P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1 \qquad .....(6)$

$(1) \quad \Rightarrow P_{00} = 3P_{01} \qquad .....(7)$

$P_{10} = P_{00} + 3P_{11} \qquad .....(8)$

$4P_{01} = P_{10} + 3Pb_1 \qquad .....(9)$

$4P_{11} = P_{01} \qquad .....(10)$

$3P_{b1} = P_{11} \qquad .....(11)$

$(7) \& (10) \Rightarrow P_{00} = 3P_{01} = 12P_{11} \qquad .....(12)$

$(12) \& (11) \Rightarrow P_{00} = 3P_{01} = 12P_{11} = 36P_{b1} \qquad .....(13)$

$(8) \qquad \Rightarrow P_{10} = P_{00} + \frac{1}{4} P_{00}$

$P_{10} = \frac{5}{4} P_{00} \qquad .....(14)$

$(6) \Rightarrow P_{00} + \frac{5}{4} P_{00} + \frac{1}{3} P_{00} + \frac{1}{12} P_{00} + \frac{1}{36} P_{00} = 1 \quad by\ (13) \& (14)$

$$P_{00} \left[1 + \frac{5}{4} + \frac{1}{3} + \frac{1}{12} + \frac{1}{36} \right] = 1$$

$$\frac{97}{36} P_{00} = 1$$

$$P_{00} = \frac{36}{97}$$

$$(13)\qquad P_{01}=\frac{1}{3}P_{00}=\frac{1}{3}\left(\frac{36}{97}\right)=\frac{12}{97}$$

$$P_{11}=\frac{1}{12}P_{00}=\frac{1}{12}\left(\frac{36}{97}\right)=\frac{3}{97}$$

$$P_{b1}=\frac{1}{36}P_{00}=\frac{1}{36}\left(\frac{36}{97}\right)=\frac{1}{97}$$

$$(14)\Rightarrow\ P_{10}=\frac{5}{4}P_{00}=\frac{5}{4}\left(\frac{36}{97}\right)=\frac{45}{97}$$

The proportion of customers entering the shop

$$=P_{00}+P_{01}=\frac{36}{97}+\frac{12}{97}=\frac{48}{97}$$

(ii) $L=P_{01}+P_{10}+2\left(P_{11}+P_{b1}\right)$

$$=\frac{12}{97}+\frac{45}{97}+2\left(\frac{3}{97}+\frac{1}{97}\right)=\frac{57}{97}+\frac{8}{97}$$

$$=\frac{65}{97}$$

(iii) $W=\dfrac{L}{\lambda\left(P_{00}+P_{01}\right)}=\dfrac{65/67}{1\left(\dfrac{36}{97}+\dfrac{12}{97}\right)}\quad=\left(\dfrac{65}{97}\right)\left(\dfrac{97}{48}\right)=\left(\dfrac{65}{48}\right)$

Problem: 21 Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates 1 and 2 are respectively 8 and 10. A customer upon completion of service at server 1 is equally likely to go to server 2 or to leave the system (i.e.,P_{11}=0,P_{12}=1/2); whereas a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise (i.e.,P_{21}=1/4,P_{22}=0). Determine the limiting probabilities, L and W.

Solution: The total arrival rates to servers 1 and 2 call them

λ_1 *and* λ_2 - Can be obtained from equation $\lambda_1=r_j+\sum_{i=1}^{k}\lambda_i P_{ij}$

That is, we have

$$\lambda_1=4+\sum_{i=1}^{2}\lambda_i P_{i1}=4+\lambda_1 P_{11}+\lambda_2 P_{21}$$

$$=4+0+\frac{1}{4}\lambda_2\qquad\left[\because P_{11}=0,P_{21}=\frac{1}{4}\right]$$

$$\lambda_1 = 4 + \frac{1}{4}\lambda_2 \qquad ......(1)$$

$$\lambda_2 = 5 + \sum_{i=1}^{2} \lambda_i \, P_{i2}$$

$$= 5 + \lambda_1 P_{12} + \lambda_2 P_{22}$$

$$= 5 + \frac{1}{4}\lambda_1 + 0 \qquad [\because P_{12} = 1/2, P_{22} = 0]$$

$$\lambda_2 = 5 + \frac{1}{2}\lambda_1 \qquad ......(2)$$

$$\therefore (1) \Rightarrow \lambda_1 = 4 + \frac{1}{4}\left[5 + \frac{1}{2}\lambda_1\right]$$

$$= 4 + \frac{5}{4} + \frac{1}{8}\lambda_1$$

$$\lambda_1 - \frac{1}{8}\lambda_1 = \frac{21}{4}$$

$$\frac{7}{8}\lambda_1 = \frac{21}{4}$$

$$\lambda_1 = \left(\frac{21}{4}\right)\left(\frac{8}{7}\right) = (3)(2) = 6$$

$$\lambda_2 = 5 + \frac{1}{2}(6) = 5 + 3 = 8$$

Given $\mu_1 = 8, \mu_2 = 0, \frac{\lambda_1}{\mu_1} = \frac{6}{8} = \frac{3}{4}, \frac{\lambda_2}{\mu_2} = \frac{8}{10} = \frac{4}{5}$

Hence P[n at server 1, m at server 2]

$$= \left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left[1 - \frac{\lambda_1}{\mu_1}\right]\left[\frac{\lambda_2}{\mu_2}\right]^{n_2}\left[1 - \frac{\lambda_2}{\mu_2}\right]$$

$$= \left(\frac{3}{4}\right)^1\left(1 - \frac{3}{4}\right)\left(\frac{4}{5}^2\right)\left(1 - \frac{4}{5}\right) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{16}{25}\right)\left(\frac{1}{5}\right) = \frac{3}{125}$$

$$L_q = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2}$$

$$= \frac{6}{8-6} + \frac{8}{10-8} = \frac{6}{2} + \frac{8}{2} = 3 + 4 = 7$$

$$W = \frac{1}{\lambda}L_q = \frac{1}{9}(7) \qquad [\because \lambda = 4 + 5]$$

$$= \frac{7}{9}$$

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Problem: 22 In a departmental store there are 2 sections, namely, grocery section and perishable (vegetables and fruits) section. Customers from outside arrive at the G-Section according to a Poisson process at a mean rate of 10/hour and they reach the P-section at a mean rate of 2/hour. The service times at both the sections are exponentially distributed with parameters 15 and 12 respectively. On finishing the job in the G-section, a customer is equally likely to go the P-section or to leave the store, whereas a customer on finishing his job in the P-section will go to the G-section with probability 0.25 and leave the store otherwise. Assuming that there is only one salesman in each section, find the probability that there are 3 customers in the G-section and 2 customers in the P-section. Find also the average number of customers in the store and the average waiting time of a customer in the store.

Solution: This system given is a Jackson's open queuing system.

$\lambda_1 \to$ total arrival rates of S_1

$\lambda_2 \to$ total arrival rates of S_2

Jackson's flow balance equations for this open model are

$$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i \; P_{ij}, j = 1,2,.....k$$

$$\lambda_1 = 10 + \sum_{i=1}^{2} \lambda_i \; P_{i1} = 10 + \lambda_i P_{11} + \lambda_i P_{21}$$

$$\lambda_1 = 10 + 0 + \frac{1}{4}\lambda_2 \qquad \left[\because P_{21} = \frac{1}{4}\right]$$

$$\lambda_1 = 10 + \frac{1}{4}\lambda_2 \qquad [P_{11} = 0] \qquad ....(1)$$

$$\lambda_2 = 2 + \sum_{i=1}^{2} \lambda_i P_{i2} = 2 + \lambda_1 P_{12} + \lambda_2 P_{22}$$

$$\lambda_2 = 2 + \frac{1}{2}\lambda_1 \qquad [\because P_{22} = 0] \qquad ....(2)$$

$$\lambda_1 = 10 + \frac{1}{4}\left[2 + \frac{1}{2}\lambda_1\right] = 10 + \frac{2}{4} + \frac{1}{8}\lambda_1$$

$$\frac{7}{8}\lambda_1 = \frac{42}{4}$$

$$\frac{7}{2}\lambda_1 = 42$$

$$\lambda_1 = \frac{42 \times 2}{7} = 12$$

$$\lambda_2 = 2 + \frac{1}{2}[12] = 8$$

Given $\mu_1 = 15, \mu_2 = 12,\ \frac{\lambda_1}{\mu_1} = \frac{12}{15} = \frac{4}{5}, \frac{\lambda_2}{\mu_2} = \frac{8}{12} = \frac{2}{3}$

P(n_1 customers in S_1 and S_2 customers in S_2)

$$P(n_1,n_2)=\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(1-\frac{\lambda_1}{\mu}\right)\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}\left(1-\frac{\lambda_2}{\mu_2}\right)$$

$$P(3,2)=\left(\frac{4}{5}\right)^{3}\left(1-\frac{4}{5}\right)\left(\frac{2}{3}\right)^{2}\left(1-\frac{2}{3}\right)$$

$$=\left(\frac{64}{125}\right)\left(\frac{1}{5}\right)\left(\frac{4}{9}\right)\left(\frac{1}{3}\right)=\frac{256}{16875}=0.0152$$

$$L=\frac{\lambda_1}{\mu_1-\lambda_1}+\frac{\lambda_2}{\mu_2-\lambda_2}=\frac{12}{15-12}+\frac{8}{12-8}=4+2=6$$

$$E(W)=\frac{1}{\lambda}L=\frac{1}{12}(6)=\frac{1}{2}[\because \lambda=10+2=12]$$

$$=\frac{1}{2}\text{hours}=\frac{1}{2}\times 60 \text{ minutes}=30 \text{ minutes.}$$

Problem: 23 Consider two servers. An average of 8 customers per hour arrive from outside at server 1 and an average of 17 customers per hour arrive from outside at server 2. Inter arrival times are exponential. Server 1 can serve at an exponential rate of 20 customers per hour and server 2 can serve at an exponential rate of 30 customers per hour. After completing service at server 1, half of the customers leave the system, and half go to server 2. After completing service at server 2, $\frac{3}{4}$ of the customers complete service, and $\frac{1}{4}$ return to server1.

(i) What fraction of the time is server 1 idle?

(ii) Find the expected number of customers at each server.

(iii) Find the average time a customer spends in the system.

(iv) How would the answers to parts (i) –(iii) change if server 2 could serve only an average of 20 customers per hour?

Solution: Given $r_1=8$

$$r_2=17$$

$$P_{12}=1/2 \quad P_{21}=1/4$$

$$P_{11}=P_{22}=0$$

Jackson's flow balance equations for this open model are

$$\lambda_j=r_j+\sum_{i=1}^{k}\lambda_j P_{ij},\, j=1,2...,k$$

$$\lambda_1=r_1+\sum_{i=1}^{2}\lambda_i P_{i1}$$

$$=8+\lambda_1 P_{11}+\lambda_2 P_{21}$$

$$=8+0+\lambda_2\frac{1}{4} \qquad [\because P_{11}=0]$$

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$$\lambda_1 = 8 + \frac{1}{4}\lambda_2 \qquad .....(1)$$

$$\lambda_2 = r_2 + \sum_{i=1}^{2} \lambda_i P_{i2}$$

$$= 17 + \lambda_1 P_{12} + \lambda_2 P_{22}$$

$$= 17 + \lambda_1 \left(\frac{1}{2}\right) + 0 \qquad [\because P_{22} = 0]$$

$$\lambda_2 = 17 + \frac{1}{2}\lambda_1 \qquad .....(2)$$

$$\therefore (1) \Rightarrow \lambda_1 = 8 + \frac{1}{4}\left[17 + \frac{1}{2}\lambda_1\right]$$

$$= 8 + \frac{17}{4} + \frac{1}{8}\lambda_1$$

$$\lambda_1 - \frac{1}{8}\lambda_1 = \frac{49}{4}$$

$$\frac{7}{8}\lambda_1 = \frac{49}{4}$$

$$\lambda_1 = \left(\frac{49}{4}\right)\left(\frac{8}{7}\right) = (7)(2) = 14$$

$$(2) \Rightarrow \lambda_2 = 17 + \frac{1}{2}(14) = 17 + 7 = 24$$

(i) Server 1 may be treated as an M/M/1/GP/∞/∞ system with $\lambda = 14, \mu = 20 \quad \rho = \frac{\lambda}{\mu} = 0.7$

$$P_0 = 1 - \rho = 1 - 0.7 = 0.3$$

Thus server 1 is idle 30% of the time.

(ii) $L = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2}$

$$= \frac{14}{20-14} + \frac{24}{30-24}$$

$$= \frac{14}{6} + \frac{24}{6} = \frac{7}{3} + 4 = \frac{19}{3}$$

(iii) $W = \frac{1}{\lambda}L = \frac{1}{25}\left(\frac{19}{3}\right) = \frac{19}{75} \quad [\because \lambda = r_1 + r_2 = 8 + 17 = 25]$

$\mu_2 = 20 < \lambda_2 = 24$. So no steady state solution exists.